Vorlesung 12.10.21

Grundintegrale

$$\int x^n dx = \begin{cases} \frac{1}{n+1} x^{n+1} & n \neq 1 \\ \ln x & n = 1 \end{cases} ; \ n \in \mathbb{R}$$

$\int e^x dx = e^x$; $\int \ln x \, dx = x \ln x - x$, siehe unten

$\int \sin x \, dx = -\cos x$; $\int \cos x \, dx = \sin x$; $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x$

Regeln Teil 2: partielle Integration

Wir wissen $(f \cdot g)' = f' \cdot g + f \cdot g' \Rightarrow f'g = (f \cdot g)' - fg' \ | \int$

$\| \int_a^b f' \cdot g \, dx = [fg]_a^b - \int f g' dx \|$ weil $\int f'(x) g(x) dx = \int \frac{d}{\cancel{dx}} (f \cdot g) \cancel{dx} - \int f(x) g'(x) dx$

$\int d(f \cdot g) = f \cdot g$

Bsp. i) $\int_2^{10} \underset{f'}{x} \underset{g}{\ln x} \, dx = [\underset{f}{\tfrac{1}{2}x^2} \cdot \underset{g}{\ln x}]_2^{10} - \int_2^{10} \underset{f}{\tfrac{1}{2}x^2} \cdot \underset{g'}{\tfrac{1}{x}} dx = [\tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2]_2^{10}$

ii) $\int \ln x \, dx = \int 1 \cdot \ln x \, dx = [x \cdot \ln x] - \int x \frac{1}{x} dx = x \ln x - x$, siehe oben

iii) $\int \underset{f'}{\sin x} \underset{g}{\cos x} \, dx = [\underset{f}{-\cos x} \underset{g}{\cos x}] - \int \underset{f}{\cos x} \underset{g'}{\sin x} \, dx \quad | + \int \cos x \sin x \, dx$

$2 \int \sin x \cos x \, dx = -\cos^2 x \Rightarrow \int \sin x \cos x \, dx = -\frac{1}{2} \cos^2 x + \text{const.}$

Regel 3: Substitution

$$\| \int_a^b f(g(x)) \cdot g'(x) dx = \int_a^b f(g(x)) \cdot \frac{dg}{dx} dx = \int_{g(a)}^{g(b)} f(g(x)) \, dg(x) \|$$

in Praxis: i) $\int_{-1}^{2} \underline{(2x+1)}^2 dx$ $\quad g = 2x+1$, $\quad \frac{dg}{dx} = 2$, $\quad \frac{1}{2} dg = dx$

Grenzen: $g(-1) = -1$, $g(2) = 5$

$= \int_{-1}^{5} g^2 \cdot \frac{1}{2} dg = \frac{1}{6} g^3 \Big|_{-1}^{5} = 21$

oder

$= \frac{1}{6} (2x+1)^3 \Big|_{-1}^{2} = \underline{\underline{21}}$

ii) im allg. $\int_{x_1}^{x_2} f(ax+b)\,dx = \int_{y_1}^{y_2} f(y)\cdot\frac{1}{a}\,dy = \frac{1}{a}\int_{y_1}^{y_2} f(y)\,dy$

$y = ax+b$

$dy = a\,dx$

$\frac{1}{a}dy = dx$

iii) unbestimmt $\int (x^2+7)^8\cdot x\,dx = \int y^8\, x\cdot\frac{dy}{2x} = \frac{1}{2}\int y^8\,dy = \frac{1}{2}\,\frac{1}{9}\,y^9 = \frac{1}{18}(x^2+7)^9$

$y = x^2+7$

$dy = 2x\,dx$

$\frac{dy}{2x} = dx$

iv) $\int \cos^5 x\,\sin x\,dx = -\int \cos^5 x\cdot\frac{\sin x}{\sin x}\,dy = -\int y^5\,dy = -\frac{1}{6}\cos^6 x$

$y = \cos x$

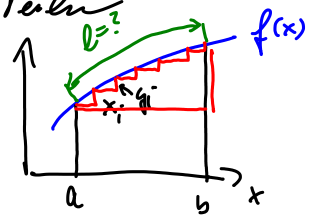
$dy = -\sin x\,dx$

$-\frac{dy}{\sin x} = dx$

Integralrechnung II – Verständnis von $\int$ und dx

Bsp. für $\int$ = Summe von vielen kleinen Teilen

Versuchen Bogenlänge auszurechnen

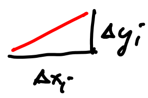


Idee 1:

$$l = \sum(\Delta x_i + \Delta y_i) = \sum\Delta x_i + \sum\Delta y_i = (b-a) + f(b) - f(a)$$

Idee 2:

$$l = \sum\sqrt{\Delta x_i^2 + \Delta y_i^2}$$



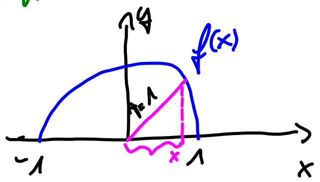
$$l = \sum\Delta x\sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}$$; z.B. alle Δx_i gleich lang

lim $\Delta \to d$; $\sum \to \int$

$$l = \int_a^b dx\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \int_a^b \sqrt{1+(f'(x))^2}\,dx \;;$$

allg. Formel für Bogenlänge

Bsp. Halbkreis $r = 1$



Pythagoras $x^2 + f(x)^2 = 1$; $f(x) = \sqrt{1-x^2}$

$$\frac{df}{dx} = \frac{d}{dx}(1-x^2)^{1/2} = \frac{1}{2}\,\frac{-2x}{\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$\Rightarrow l = \int_{-1}^{1} dx\sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2} = \int_{-1}^{1} dx\sqrt{1+\frac{x^2}{1-x^2}} = \int_{-1}^{1} dx\,\frac{1}{\sqrt{1-x^2}}$$

NR: Subst.
$x = \sin y$
$dx = \cos y \, dy$

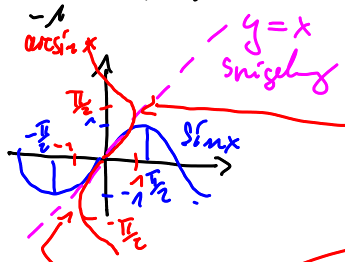
$$\Rightarrow \int dx \frac{1}{\sqrt{1-x^2}} = \int dy \frac{\cos y}{\sqrt{1-\sin^2 x}} \quad ; \quad \sin^2 y + \cos^2 y = 1$$

$$1 - \sin^2 y = \cos^2 y$$

$\int 1 \cdot dy = y$; y aus $x = \sin y$
$y = \arcsin x$

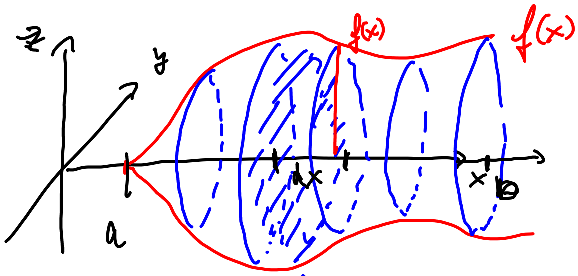
$$\int dx \frac{1}{\sqrt{1-x^2}} = \arcsin x$$

$$\Rightarrow \frac{l_{Kreis}}{2} = \int_{-1}^{1} dx \frac{1}{\sqrt{1-x^2}} = \arcsin x \Big|_{-1}^{1} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$



Kreisumfang $U = 2\pi r$

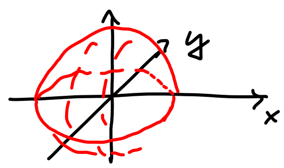
Bsp. 2 Volumen von Rotationskörper



$$V = \int dV = \int_a^b dx \cdot \pi \cdot f(x)^2$$

$$dV = \text{Grundfläche} \cdot \text{Höhe} = \pi \cdot f(x)^2 \cdot dx$$

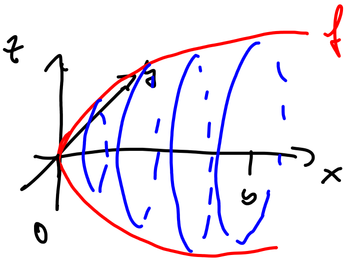
Bsp. i) V-Kugel



$$f(x) = \sqrt{r^2 - x^2} \Rightarrow V = \int_{-r}^{r} \pi \left(\sqrt{r^2-x^2}\right)^2 dx = \int_{-r}^{r} \pi (r^2 - x^2) dx = \left[\pi r^2 \cdot x - \frac{1}{3}x^3\right]_{-r}^{r}$$

$$= \frac{4}{3}\pi r^3$$

ii) Paraboloid

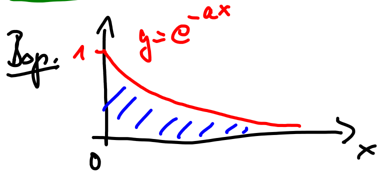


$f(x) = \sqrt{x}$

$$V = \int_0^b dx \, \pi \left(\sqrt{x}\right)^2 dx = \int_0^b dx \, \pi \cdot x = \frac{\pi}{2} b^2$$

Integrale mit Grenzen an $\pm\infty$

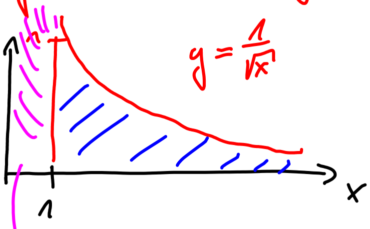
Bsp.



$$A=\int_0^\infty e^{-ax}\,dx \underset{-ax=y}{=} -\frac{1}{a}\int_0^{-\infty} e^{y}\,dy = \left[-\frac{1}{a}e^{y}\right]_0^{-\infty} = 0-\left(-\frac{1}{a}\right) = \frac{1}{a}$$

Es kann passieren, dass $\int$ nicht existiert obwohl $f(x\to\infty)\to 0$

Bsp.



$$A=\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \int_1^\infty x^{-\frac{1}{2}}\,dx = \frac{1}{-\frac{1}{2}+1}\,x^{-\frac{1}{2}+1}\Big|_1^\infty = 2\sqrt{x}\Big|_1^\infty = \infty$$

aber $A=\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_0^1 = 2$

Integrale an Polstellen z.B.

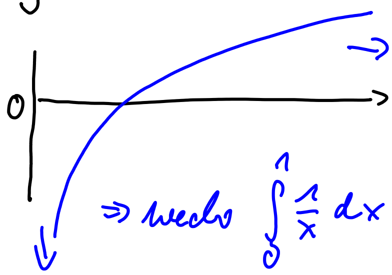
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$$

aber $\int_0^1 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_0^1 \to \infty$ ≙ existiert nicht

jedoch $\int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = 0-\left(-\frac{1}{1}\right) = 1$

am schlimmsten ist $\frac{1}{x}$

$$\int \frac{1}{x}\,dx = \ln x$$



$\Rightarrow$ weder $\int_0^1 \frac{1}{x}\,dx$ noch $\int_1^\infty \frac{1}{x}\,dx$ existieren