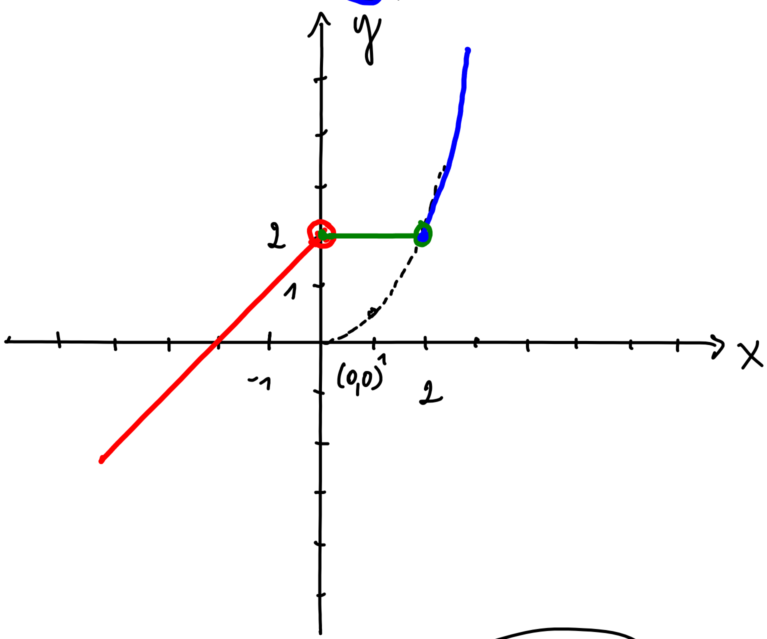
Vorrechnen 4 07.10.22

Aufgabe 1: Abschnittsweise definierte Funktionen

a) $f(x) = \begin{cases} x+2, & x<0 \\ 2, & 0 \le x < 2 \\ \frac{x^2}{2}, & x \ge 2 \end{cases}$

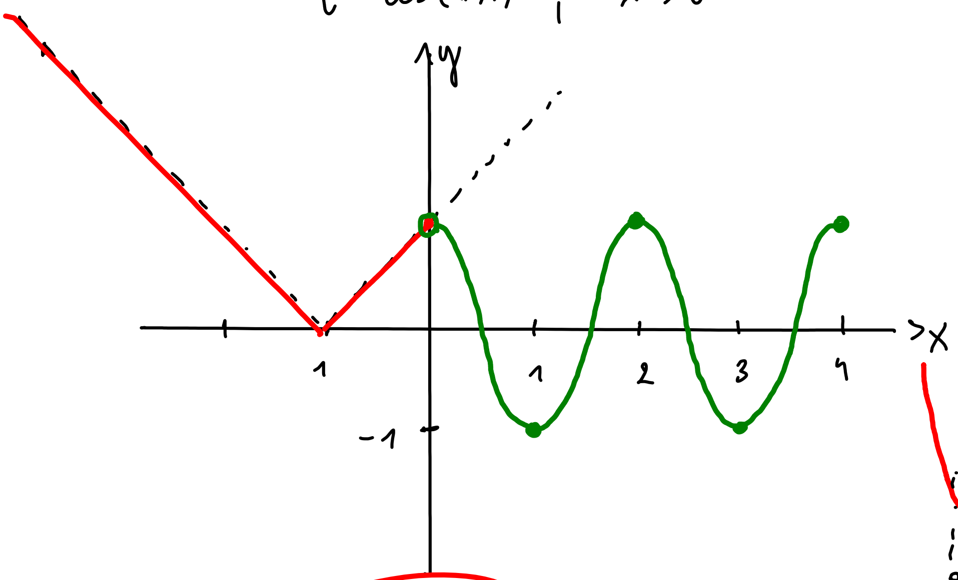


$\lim_{x \to -\infty} f(x) = -\infty$

$\lim_{x \to +\infty} f(x) = +\infty$

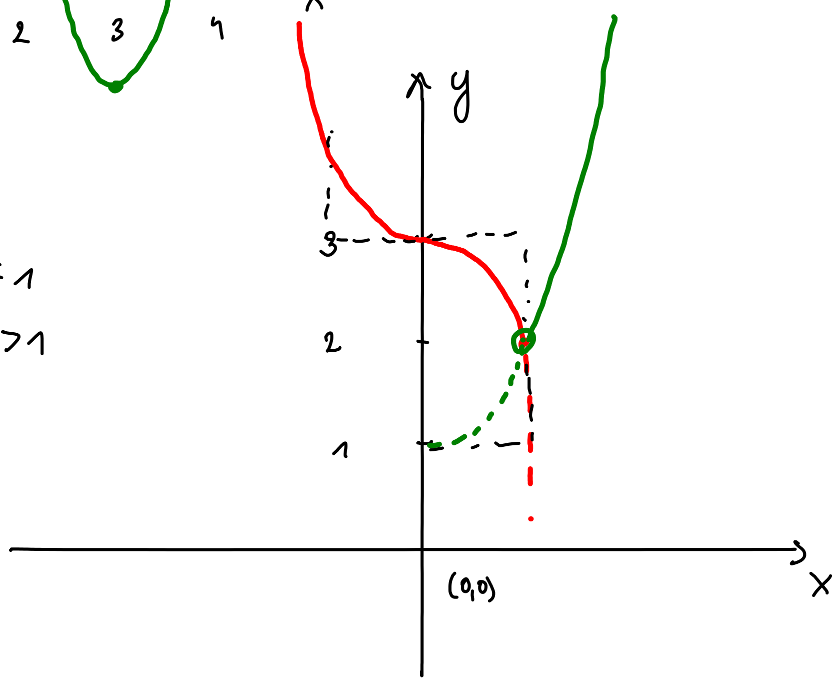
$D = \mathbb{R}, \quad W = \mathbb{R}$

b) $g(x) = \begin{cases} |-x-1|, & x \le 0 \\ \cos(\pi x), & x > 0 \end{cases}$ $\to |x+1|$



$D = \mathbb{R}, \quad W = [-1, +\infty)$

c) $h(x) = \begin{cases} -x^3+3, & x \le 1 \\ x^2+1, & x > 1 \end{cases}$



$D = \mathbb{R}, \quad W = [2, +\infty)$

Aufgabe 2: Zusammengesetzte und verkettete Funktionen

(i) $f+g, \quad f-g, \quad f\cdot g, \quad \frac{f}{g}$

a) $f(x) = x-2, \quad g(x) = 1-2x$

$f+g = x-2+1-2x = -x-1, \quad (f+g)(x): \mathbb{R} \to \mathbb{R}$

$f-g = x-2-1+2x = 3x-3, \quad (f-g)(x): \mathbb{R} \to \mathbb{R}$

$f\cdot g = (x-2)(1-2x) = x-2x^2-2+4x = -2x^2+5x-2$

$x^2 - \frac{5}{2}x$

$-2x^2+5x-2 = -2\cdot(x^2 - \frac{5}{2}x) - 2$

$(x+a/2)^2 = x^2 + ax + \frac{a^2}{4} \Rightarrow x^2$

$a = -\frac{5}{2} \Rightarrow x^2 - \frac{5}{2}x = (x-\frac{5}{4})^2 - \frac{25}{16}$

$(f\cdot g)(x) = -2\cdot\left((x-\frac{5}{4})^2 - \frac{25}{16}\right) - 2$

$= +\frac{25}{8} - 2(x-\frac{5}{4})^2 - 2$

$= \frac{25}{8} - 2 - 2(x-5/4)^2$

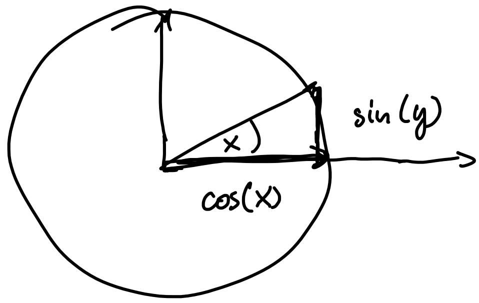
$\frac{25}{8} - \frac{16}{8} = \frac{9}{8} \qquad W = (-\infty, \frac{9}{8}]$

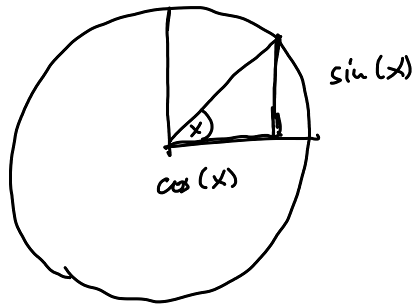
$\lim_{x\to\pm\infty} -2x^2+5x-2 = -\infty$

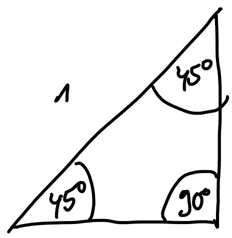
$\frac{f}{g} = \frac{x-2}{1-2x} \qquad D = \mathbb{R} \setminus \{1/2\}$

b) $f(x) = \sin(x), \quad g(x) = \cos(x)$

$(f+g)(x) = \sin(x) + \cos(x) \qquad D = \mathbb{R}, \quad W = [-\sqrt{2}, \sqrt{2}]$







$\leftarrow \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\uparrow$

$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$(f-g)(x) = \sin(x) - \cos(x) \qquad D = \mathbb{R},$

$f \cdot g\,(x) = \sin(x) \cdot \cos(x) \qquad D = \mathbb{R}$

$f/g\,(x) = \frac{\sin(x)}{\cos(x)} \equiv \tan(x) \qquad D = \mathbb{R} \setminus \{\frac{\pi}{2} + k \cdot \pi \mid k \in \mathbb{Z}\}$

$\frac{\pi}{2} + k \cdot \pi, \quad k \in \mathbb{Z}$

c) $f(x) = x^2 - 1, \quad g(x) = x - 1$

$f+g = x^2 - 1 + x - 1 = x^2 + x - 2$

$D = \mathbb{R}$

$f-g = x^2 - 1 - x + 1 = x^2 - x \qquad D = \mathbb{R}$

$f \cdot g = (x^2 - 1)(x - 1) = x^3 - x^2 - x + 1, \quad D = \mathbb{R}$

$\lim_{x \to \pm\infty} f \cdot g = \pm\infty$

$$f/g = \frac{x^2-1}{x-1} \qquad D = \mathbb{R} \setminus \{1\} \qquad \frac{(x+1)(x-1)}{x-1} \to x+1$$

d) $f(x) = \sqrt{1+x}\,, \quad g(x) = \sqrt{x}$

$$f(x) + g(x) = \sqrt{1+x} + \sqrt{x} \qquad D = \mathbb{R}^+$$

$$f(x) - g(x) = \sqrt{1+x} - \sqrt{x} \qquad D = \mathbb{R}^+$$

$$f \cdot g = \sqrt{1+x} \cdot \sqrt{x}\,, \qquad ?$$

$$D = \mathbb{R} \setminus [-1, 0]$$

$$D = [0, 1)$$

$$f \cdot g = \sqrt{x + x^2} \qquad x + x^2 \geq 0 \quad ?$$

$$x \cdot (1+x) \geq 0$$

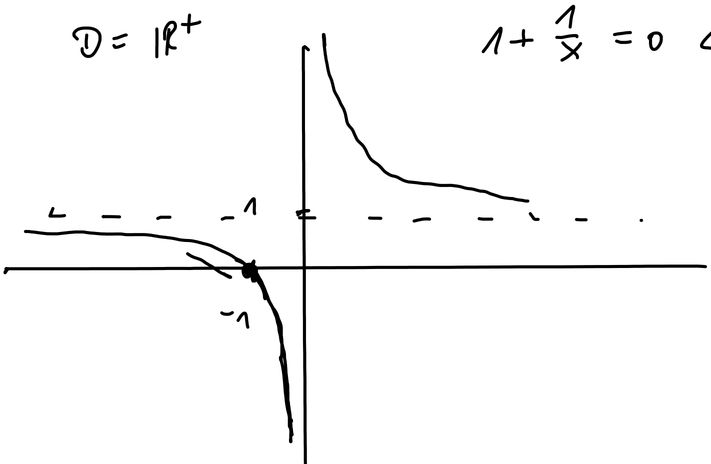
Scheitelform: $x^2 + x = \underbrace{(x + 1/2)^2}_{x^2 + x + 1/4} - 1/4$

$$(x + 1/2)^2 = 1/4 \Rightarrow x + 1/2 = \pm 1/2$$

$$x = 0 \vee x = -1$$

$$\frac{f}{g} = \frac{\sqrt{x+1}}{\sqrt{x}} = \sqrt{\frac{x+1}{x}} = \sqrt{1 + \frac{1}{x}} \qquad x \in [-1, 0]$$

$\uparrow$ $D = \mathbb{R}^+$ $\qquad \uparrow$ $1 + \frac{1}{x} = 0 \Leftrightarrow x + 1 = 0 \Rightarrow x = -1$



$$1 + \frac{1}{x} < 0 \text{ für } x \in (-1, 0)$$

$$D = \mathbb{R} \setminus (-1, 0)$$

(ii) a) $h(x) = \ln(x+1)$ $\qquad h(x) = g(f(x))$

$f(x) = x+1, \quad g(x) = \ln(x)$

b) $h(x) = \left(\frac{x+2}{x+1}\right)^2, \qquad f(x) = \frac{x+2}{x+1}, \quad g(x) = x^2$

c) $h(x) = \cos^2(x)$ $\qquad \sin x \quad \cos x \quad \ln x$

$f(x) = \cos(x), \quad g(x) = x^2$

(iii) $f(x) = x^2, \quad g(x) = \sqrt{x}, \quad h(x) = 1/x$

a) $g \circ f = g(f(x)) = \sqrt{x^2} = |x| \qquad D = \mathbb{R}$

$f \circ g = f(g(x)) = \sqrt{x}^2 = |x| = x \qquad D = \mathbb{R}_0^+$

b) $f \circ (g+h) = (\sqrt{x} + 1/x)^2 \qquad D = \mathbb{R}^+$

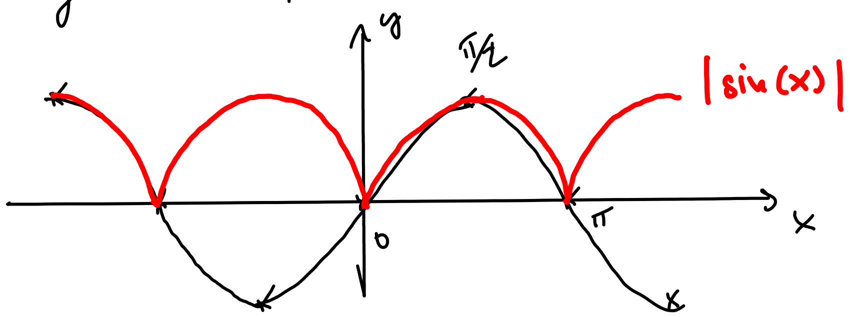
$x + 2\frac{\sqrt{x}}{x} + 1/x^2 = x + \frac{2}{\sqrt{x}} + 1/x^2$

c) $h \circ (f \cdot g) = \frac{1}{x^2 \cdot \sqrt{x}} \qquad D = \mathbb{R}^+ \qquad \frac{1}{x^2 \cdot \sqrt{x}} = \frac{1}{x^2 \cdot x^{1/2}} = \frac{1}{x^{5/2}}$

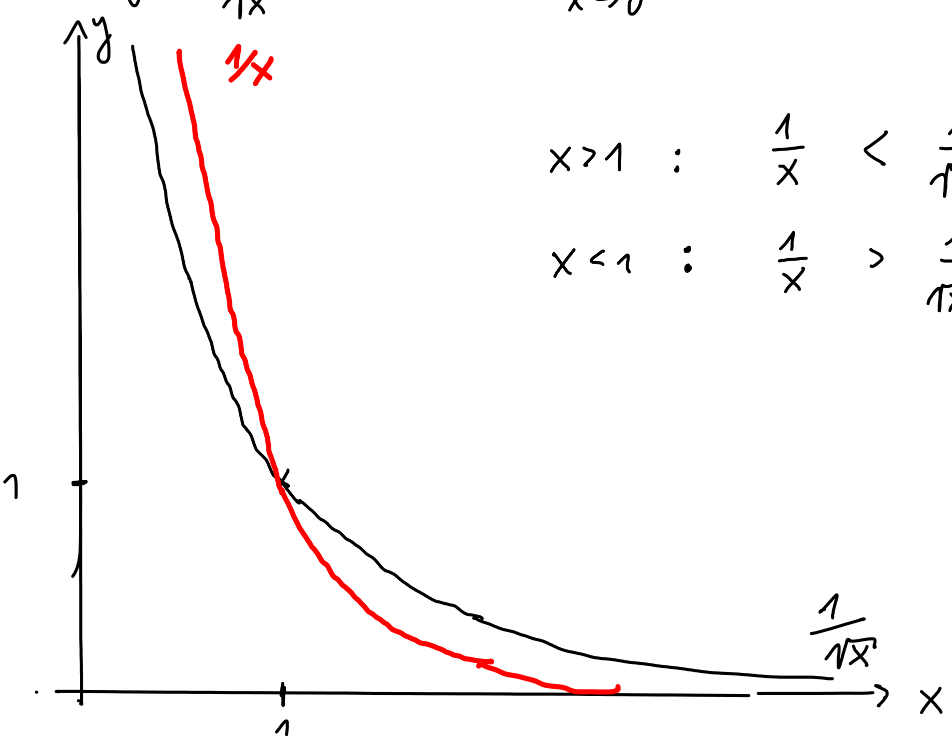
d) $f \circ (g \circ h) = f \circ g(h(x)) = \underbrace{(\sqrt{1/x})}_{\mathbb{R}^+}{}^2 = (1/x = \mathbb{R} \setminus \{0\})$

Aufgabe 3 Kurvendiskussion $\qquad x^{5/2} = (\sqrt{x})^5 = \sqrt{x^5}$

(i) a) $y = |\sin(x)|$



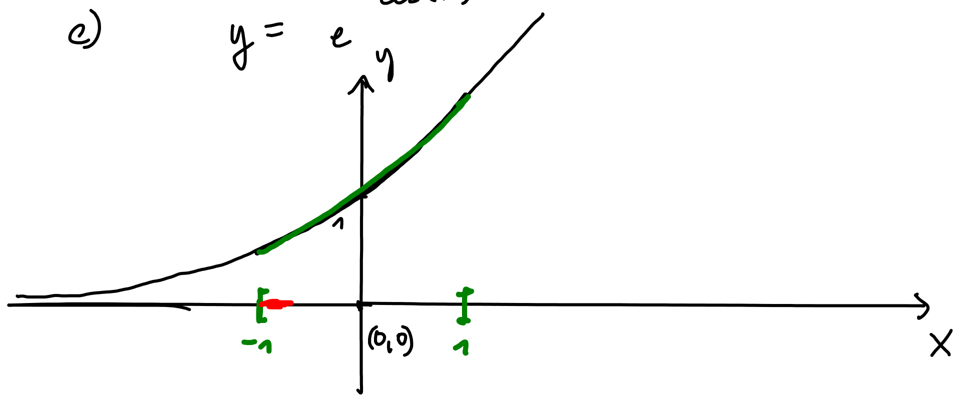
b) $y = \frac{1}{\sqrt{x}}$ $D = \mathbb{R}^+$ $\lim_{x \to 0^+} \frac{1}{\sqrt{x}} = +\infty$ $\lim_{x \to +\infty} \frac{1}{\sqrt{x}} = 0^+$

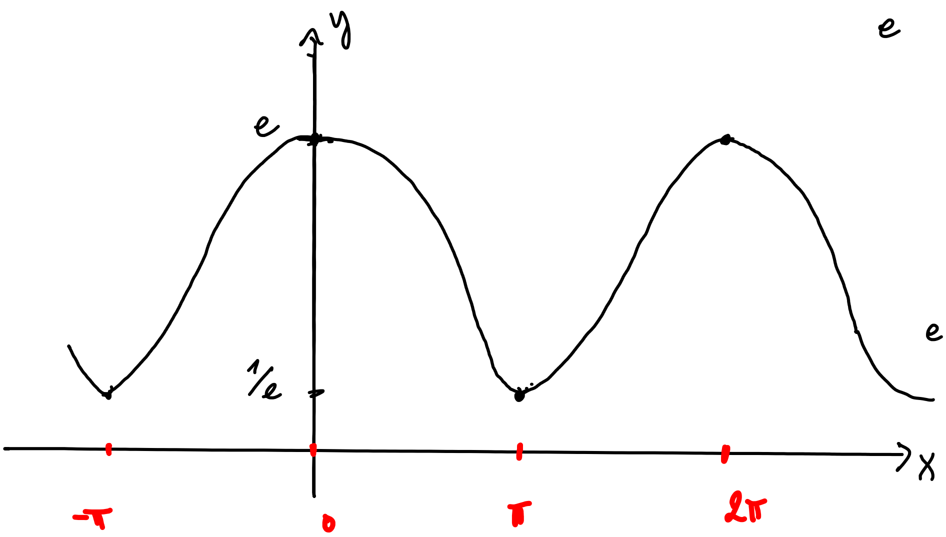


$x > 1 : \quad \frac{1}{x} < \frac{1}{\sqrt{x}}$

$x < 1 : \quad \frac{1}{x} > \frac{1}{\sqrt{x}}$

c) $y = e^{\cos(x)}$





$e^{\cos(x)}$ für $\cos(x)$ maximal

$\cos(x) = 1$

$\rightarrow e^1$

$e^{\cos(x)}$ $\cos(x) = -1$

$\rightarrow 1/e$

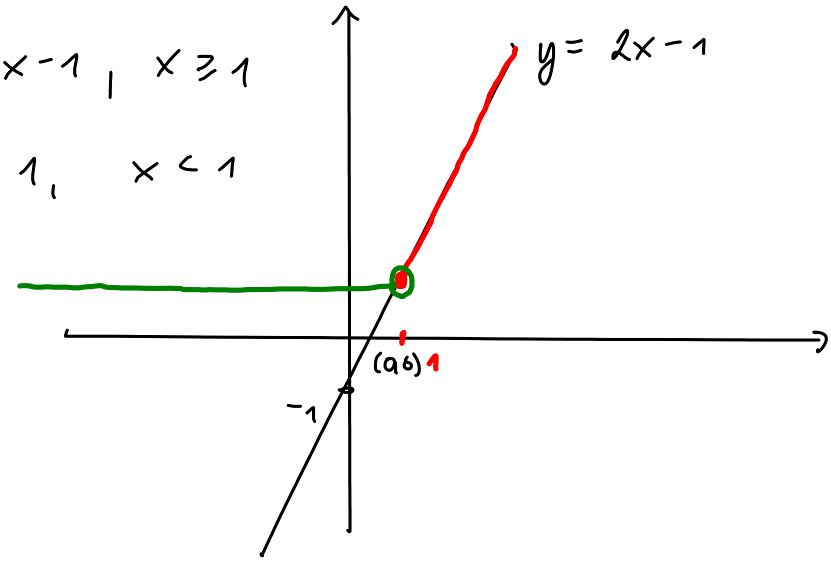
$x - 1 \leq 0$

$\Rightarrow x \leq 1$

d) $y = x + |x-1|$ $|x-1|$ $D = \mathbb{R}$

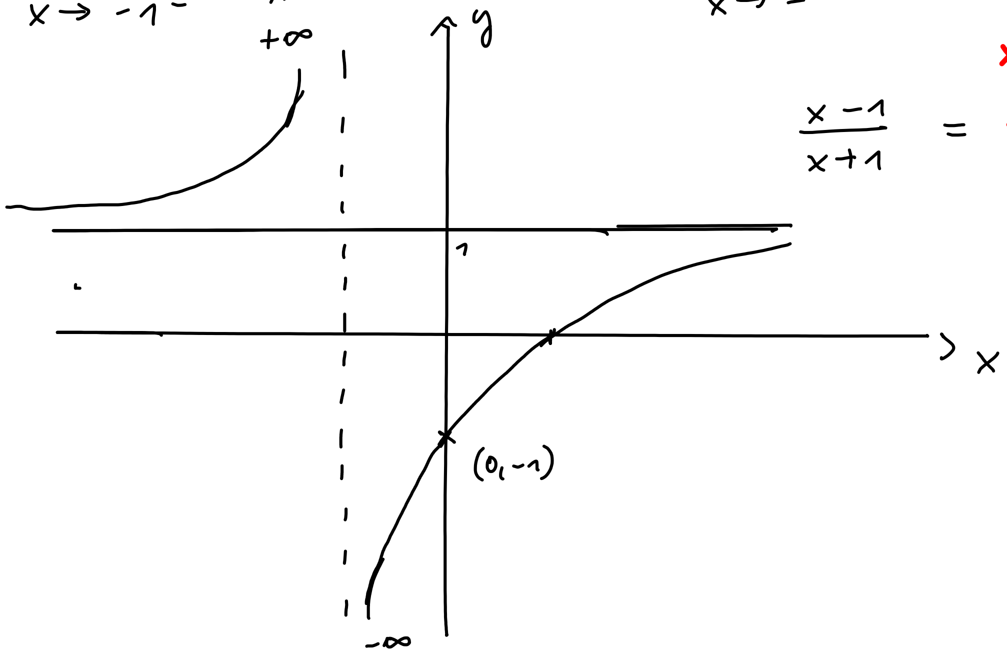
$$|x-1| = \begin{cases} x-1, & x \geq 1 \\ -x+1, & x < 1 \end{cases}$$

$$y = \begin{cases} x + x - 1 = 2x - 1, & x \geq 1 \\ x - x + 1 = 1, & x < 1 \end{cases}$$



e) $y = \dfrac{x-1}{x+1}$ $D = \mathbb{R} \setminus \{-1\}$

$$\lim_{x \to -1^{\pm}} \frac{x-1}{x+1} = \mp\infty \qquad \lim_{x \to \pm\infty} \frac{x-1}{x+1} = 1^{\pm}$$



$x \neq 0$

$$\frac{x-1}{x+1} = \frac{\cancel{x}(1 - \overset{\to 0}{1/x})}{\cancel{x}(1 + \underset{\to 0}{1/x})}$$

f) $y = \ln(4 - x^2)$ $D = (-2, 2)$

Definitionsbereich $\ln(x)$: $\mathbb{R}^+$

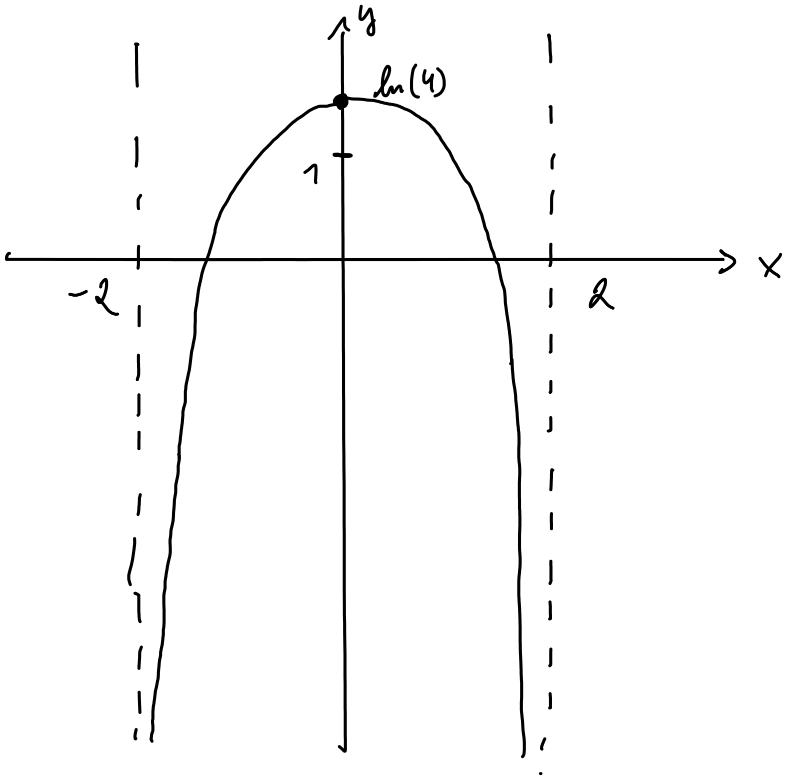
$$4 - x^2 > 0 \Rightarrow 4 > x^2 \Rightarrow x \in (-2, 2)$$

$$\lim_{x \to 0^+} \ln(x) = -\infty \qquad \lim_{x \to \infty} \ln(x) = \infty$$

$$\lim_{x \to -2^+} \ln(4 - x^2) = -\infty \qquad \lim_{x \to +2^-} \ln(4 - x^2) = -\infty$$

$\ln(4) = y(0)$ $\qquad \ln(1) = 0$

$4 - x^2 = 1 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$



(ii) a) $\frac{5}{6} \sin(7x+8) = f(x)$

$-1 \leq \sin(x) \leq 1 \Rightarrow -\frac{5}{6} \leq f(x) \leq 5/6$

Periode: $\frac{2\pi}{7}$ $\qquad \sin\left(7\left(x + \frac{2\pi}{7}\right) + 8\right)$

$= \sin(7x + 8 + 2\pi) = \sin(7x+8)$

b) $f(x) = \frac{1}{4 + \sin(x)}$ $\qquad \frac{1}{5} \leq f(x) \leq \frac{1}{3}$

$-1 \leq \sin(x) \leq 1 \rightarrow \frac{1}{4+1}, \frac{1}{4-1}$

Periode: 2π

(iii) $f(x) = x^3 - 39x - 70$

durch raten: $x = -2 \rightarrow (x+2) \cdot g(x) = f(x)$

$$(x^3 - 39x - 70) : (x+2) = x^2 - 2x - 35$$
$$-(x^3 + 2x^2)$$
$$-2x^2 - 39x - 70$$
$$-(-2x^2 - 4x)$$
$$-35x - 70$$
$$-(-35x - 70)$$
$$0$$

$$f(x) = (x+2)(x^2 - 2x - 35)$$

$$x^2 - 2x - 35 = 0$$

$$\Rightarrow x = 1 \pm \underbrace{\sqrt{1+35}}_{6} = 7 \vee -5$$

$$f(x) = (x+2)(x+5)(x-7)$$

Minima und Maxima

$$f'(x) = 3x^2 - 39 \overset{!}{=} 0 \Rightarrow x^2 = \frac{39}{3} = 13$$

$$x = \pm\sqrt{13}$$

$$f''(x) = 6x \qquad f''(\sqrt{13}) > 0 \qquad f''(-\sqrt{13}) < 0$$

$\hookrightarrow$ Minimum $\qquad$ $\hookrightarrow$ Maximum

Wendepunkt: $f''(x) = 0$

$\Rightarrow x = 0$