Vorrechnen 5: Trigonometrische Funktionen

Aufgabe 1: Graphen trigonometrischer Funktionen

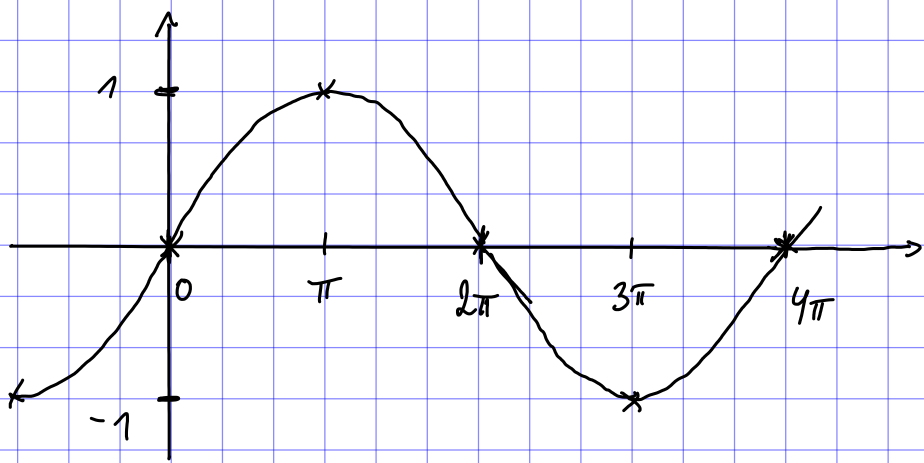
a) $\sin(x/2)$

Nullstellen: $x/2 = z \cdot \pi, \; z \in \mathbb{Z} \Rightarrow x = 2\pi \cdot z$

Maxima: $x/2 = \pi/2 + z \cdot 2\pi, \; z \in \mathbb{Z} \Rightarrow x = \pi + 4\pi z$

$= (4z+1) \cdot \pi$

Streckung in x-Richtung um Faktor 2



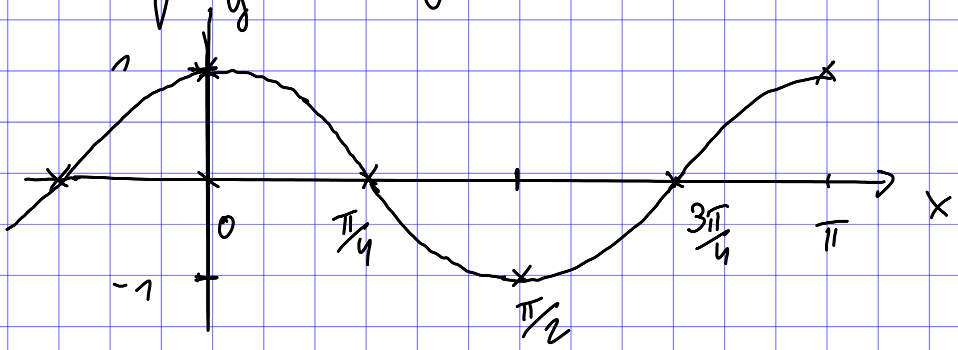
b) $\cos(2x)$ Nullstellen: $\pi/2 + z \cdot \pi, \; z \in \mathbb{Z}$

$\rightarrow 2x = \pi/2 + z \cdot \pi$

$\Rightarrow x = \pi/4 + z \cdot \pi/2 = \frac{\pi}{2}(\frac{1}{2} + z)$

Maxima $2\pi z = 2x \Rightarrow x = \pi z$

Stauchung in x-Rtg um $1/2$



c) $\sin\left(\frac{x}{3} - 3\right)$

Nullstellen: $\frac{x}{3} - 3 = z \cdot \pi \, , \; z \in \mathbb{Z}$

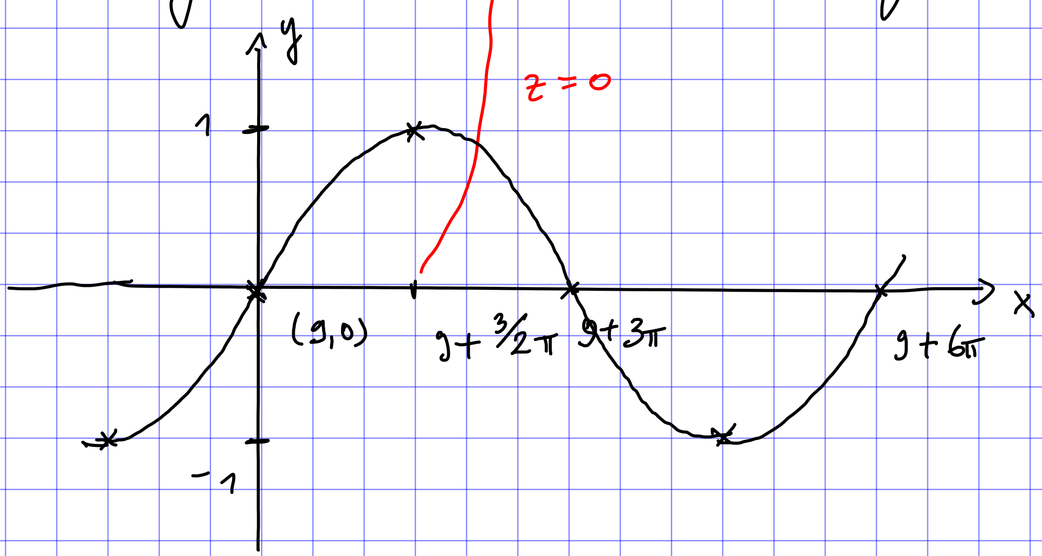
$$\Rightarrow x = 3 \cdot (z \cdot \pi + 3) = 3\pi z + 9$$

Maxima: $\frac{x}{3} - 3 = \frac{\pi}{2} + z \cdot 2\pi$

$$\Rightarrow x = \left(6z + \frac{3}{2}\right)\pi + 9$$

$$\sin\left(\frac{x}{3} - 3\right) = \sin\left(\frac{1}{3}(x - 9)\right)$$

Verschiebung um 9 nach rechts, Stauchung um 1/3



d) $\sin(3x) - \frac{1}{2}$

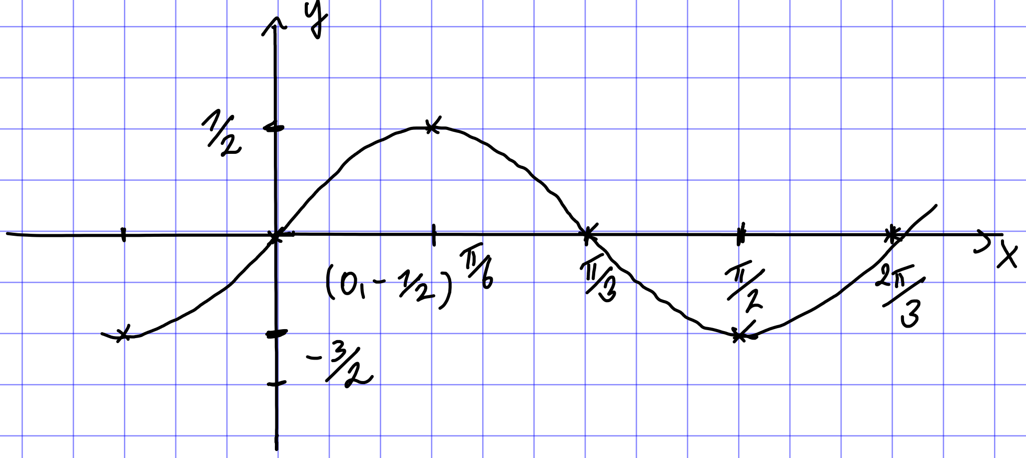
Maxima: $\frac{\pi}{2} + z \cdot 2\pi = 3x \Rightarrow x = \frac{\pi}{6} + z \cdot \frac{2}{3}\pi$

$$= \frac{\pi}{3} \cdot \left(\frac{1}{2} + 2z\right)$$

$-\frac{1}{2}$ - Stellen: $3x = z \cdot \pi \, , \; z \in \mathbb{Z}$

$$x = z \cdot \frac{\pi}{3}$$

Stauchung um 3 in x-Rtg und Verschiebung um 1/2 in neg. y-Rtg



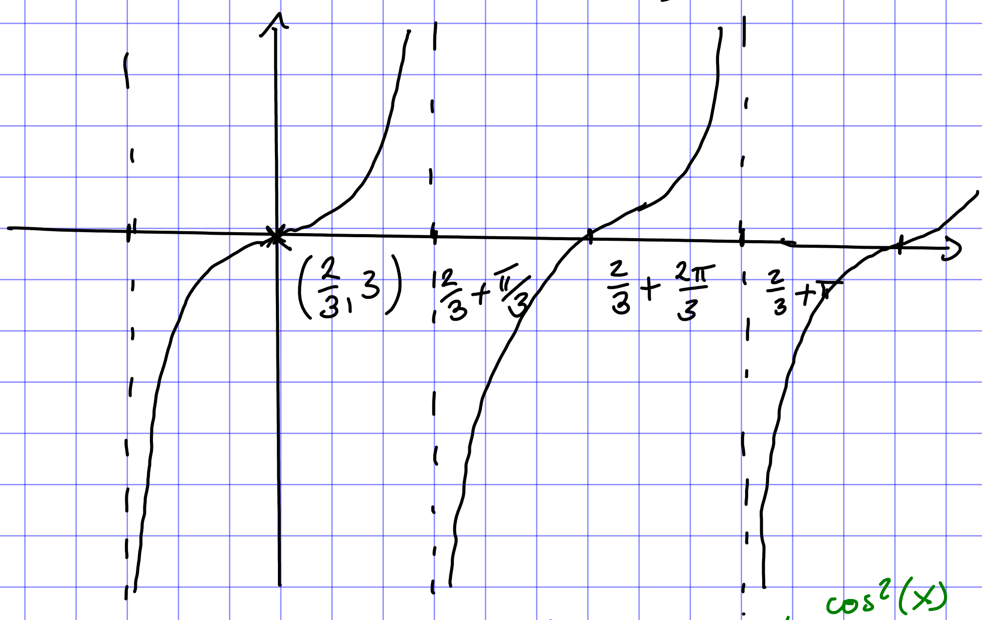
e) $\tan\left(\frac{3}{2}x - 1\right) + 3$

Nullstellen: $\frac{3}{2}x - 1 = z \cdot \pi, \quad z \in \mathbb{Z} \Rightarrow x = \frac{2}{3}(z\pi + 1)$
(3 - Stellen)

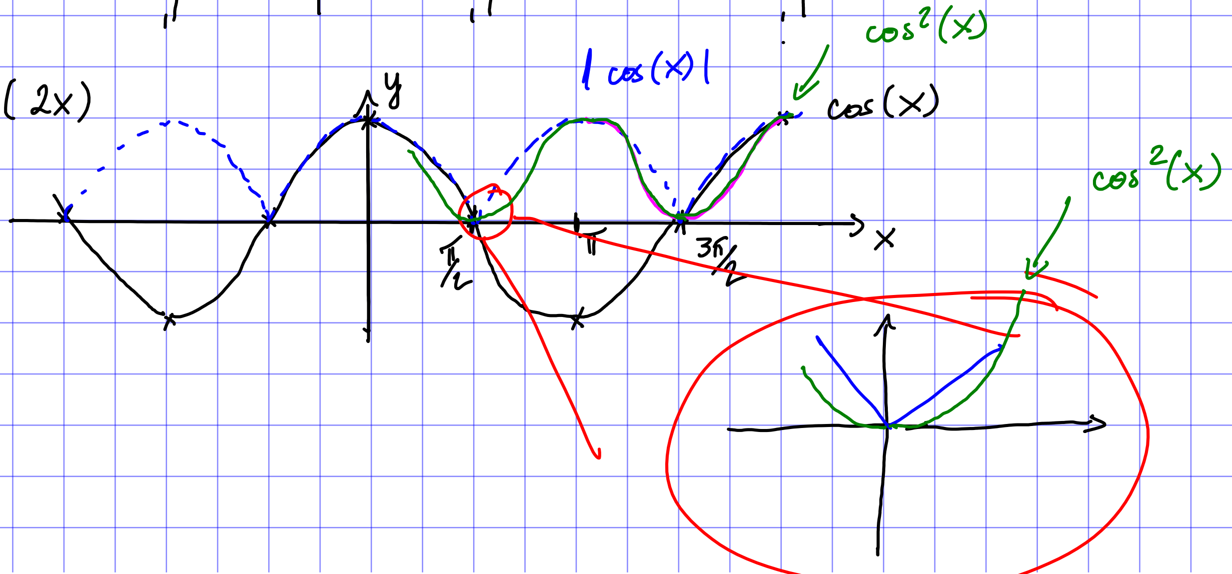
Polstellen: $\frac{3}{2}x - 1 = \frac{\pi}{2} + z \cdot \pi, \quad z \in \mathbb{Z} \Rightarrow x = \frac{2}{3}\left(z\pi + 1 + \frac{\pi}{2}\right)$

$$= \frac{2}{3} + \frac{\pi}{3}(2z + 1)$$

$$\tan\left(\frac{3}{2}x - 1\right) = \tan\left(\frac{3}{2}\left(x - \frac{2}{3}\right)\right)$$



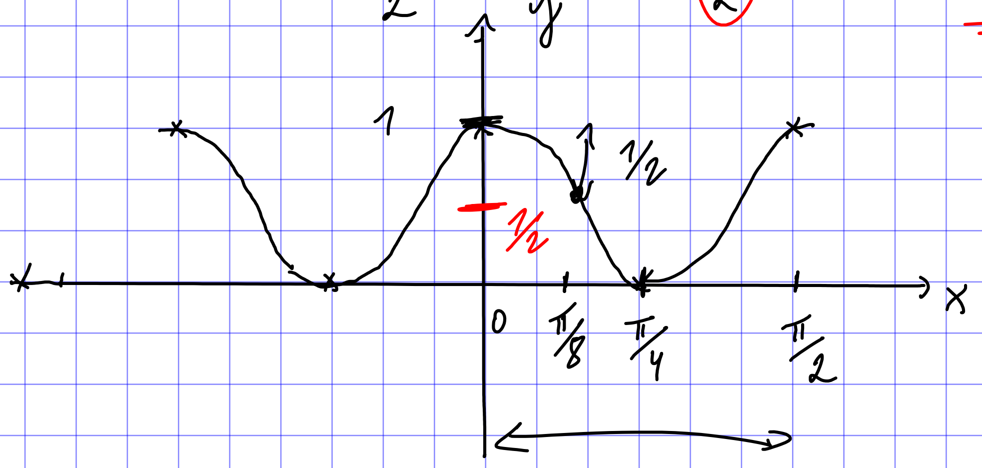
f) $\cos^2(2x)$



$$\cos^2(2x) = \tfrac{1}{2} + \tfrac{1}{2}\cos(4x)$$

denn :
$$\cos(4x) = \cos(2x + 2x)$$
$$= \cos(2x)\cos(2x) - \sin(2x)\sin(2x)$$
$$= \cos^2(2x) - \underbrace{\sin^2(2x)}_{= 1 - \cos^2(2x)}$$
$$= 2\cos^2(2x) - 1$$

$$\cos^2(2x) = \frac{\cos(4x)+1}{2} = \frac{1}{2}\cos(4x) + \tfrac{1}{2}$$



Aufgabe 2: Additionstheoreme

(i) a) $\sin(\pi/4)\cos(\pi/4) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$

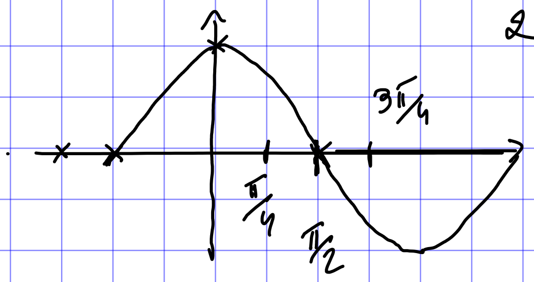
$$2\sin(\pi/4)\cos(\pi/4) = \sin(\pi/4 + \pi/4) = \sin(\pi/2) = 1$$
$$\Rightarrow \sin(\pi/4)\cos(\pi/4) = \tfrac{1}{2}$$

b) $\cos(2a) = \cos(a)\cos(a) - \underbrace{\sin(a)\sin(a)}_{\sin^2(a) = 1 - \cos^2(a)}$

$$= 2\cos^2(a) - 1$$

c) $\cos^2(3\pi/8)$, $\cos^2(a) = \frac{\cos(2a)+1}{2}$

$$\Rightarrow \cos^2\left(\tfrac{3\pi}{8}\right) = \frac{\cos\left(\tfrac{3\pi}{4}\right)+1}{2}$$



$$\cos\left(\tfrac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\cos^2\left(\tfrac{3\pi}{8}\right) = \frac{1-\tfrac{1}{\sqrt{2}}}{2} = \frac{1}{2} - \frac{1}{\sqrt{8}}$$

d) $\underline{1-2\sin^2(\pi/8)}$

$$\frac{1}{\sqrt{2}} = \underline{\cos(\pi/4)} = \cos\left(\pi/8+\pi/8\right) = \underbrace{\cos\pi/8\,\cos\pi/8}_{} - \sin\pi/8\,\sin\pi/8$$

$$\cos^2\pi/8 = 1-\sin^2\pi/8$$

$$= \underline{1-2\sin^2\pi/8} = \frac{1}{\sqrt{2}}$$

e) $\dfrac{1}{\tan^2\pi/8+1}$ $\qquad \tan^2x+1 = \dfrac{1}{\cos^2x}$

$$= \frac{1}{\frac{1}{\cos^2\pi/8}} = \cos^2\pi/8$$

$$\cos\pi/4 = \cos(\pi/8+\pi/8)$$

$$= \cos^2\pi/8 - \sin^2\pi/8$$

$$= 2\cos^2\pi/8 - 1$$

$$\Rightarrow \cos^2\pi/8 = \frac{\cos\pi/4+1}{2} = \frac{1+\frac{1}{\sqrt{2}}}{2} = \frac{1}{2}+\frac{1}{\sqrt{8}}$$

f) $\cot^2\left(3\pi/8\right)-1$

$$= \frac{\cos^2\left(3\pi/8\right)}{\sin^2\left(3\pi/8\right)} - 1$$

$$\cos^2\left(\tfrac{3\pi}{8}\right) = \frac{1}{2}-\frac{1}{\sqrt{8}} \qquad \sin^2\left(\tfrac{3\pi}{8}\right) = 1-\cos^2\left(\tfrac{3\pi}{8}\right)$$

$$\Rightarrow \sin^2(3\pi/8) = 1/2 + \frac{1}{\sqrt{8}}$$

$$\cot(3\pi/8) - 1 = \underbrace{\frac{1/2 - 1/\sqrt{8}}{1/2 + 1/\sqrt{8}}}_{\cdot 4} - 1 = \frac{2-\sqrt{2}}{2+\sqrt{2}} - 1$$

$$= \frac{2-\sqrt{2} - 2 - \sqrt{2}}{2+\sqrt{2}} = \frac{-2\sqrt{2}}{2+\sqrt{2}} \begin{array}{l} |\cdot(2-\sqrt{2}) \\ |\cdot(2-\sqrt{2}) \end{array}$$

$$= \frac{-2\sqrt{2}(2-\sqrt{2})}{4-2} = -\sqrt{2}\cdot(2-\sqrt{2}) = -2\sqrt{2}+2$$

(ii) a) $\tan(x \pm y) = \dfrac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$

$$\frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)} = \frac{\frac{\sin(x)}{\cos(x)} \pm \frac{\sin(y)}{\cos(y)}}{1 \mp \frac{\sin(x)\sin(y)}{\cos(x)\cos(y)}} \quad \Big| \cdot \cos x \cos y$$

$$= \frac{\sin(x)\cos(y) \pm \sin(y)\cos(x)}{\cos(x)\cos(y) \mp \sin(x)\sin(y)} \begin{array}{l} \} \sin(x \pm y) \\ \} \cos(x \pm y) \end{array} = \tan(x \pm y)$$

b) $\cot(x \pm y) = \dfrac{\cot(x)\cot(y) \mp 1}{\cot(y) \pm \cot(x)} \begin{array}{l} | : (\cot(x)\cdot\cot(y)) \\ | : (\cot(x)\cdot\cot(y)) \end{array}$

$$= \frac{1 \mp \overset{\tan(x)}{\frac{1}{\cot(x)}} \overset{\tan(y)}{\frac{1}{\cot(y)}}}{\frac{1}{\cot(x)} \pm \frac{1}{\cot(y)}} = \frac{1 \mp \tan(x)\tan(y)}{\tan(x) \pm \tan(y)}$$

$$\frac{1}{\tan(x \pm y)}$$

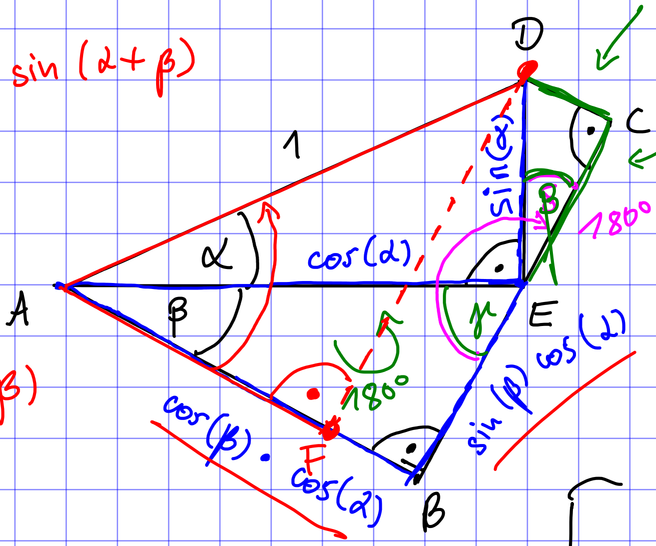
$$= \cot(x \pm y)$$

Aufgabe 3: Geometrie der Additionstheoreme

$$\frac{FD}{\underbrace{AD}_{=1}} = \sin(\alpha+\beta)$$

$\sin\beta \cdot \sin\alpha$

$\cos\beta \cdot \sin\alpha$



$\mu + 90° = \delta \Rightarrow \mu = 90° - \delta$

$\mu = 180° - 90° - \beta$

$= 90° - \beta$

$\Rightarrow \delta = \beta$

$$\frac{AF}{1} = \cos(\alpha+\beta)$$

AED:

$AE = \cos(\alpha)$

$ED = \sin(\alpha)$

ABE: AE: Hypotenuse

AB: Ankathete

$$\frac{AB}{AE} = \cos(\beta)$$

$\Rightarrow AB = \cos(\beta) \cdot AE$

ECD:

$EC = \cos(\beta) \cdot \sin(\alpha)$

$DC = \sin(\beta) \cdot \sin(\alpha)$

BE: Gegenkathete

$BE = \sin(\beta) \cdot \underbrace{\cos(\alpha)}_{= AE}$

(1) $DF = BC = \cos\alpha \sin\beta + \cos\beta \sin\alpha \overset{AFD}{=} \sin(\alpha+\beta)$

(2) $AF = \cos(\alpha+\beta) = AB - \underbrace{FB}_{DC} = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$