Vorrechnen 6: Vektoralgebra

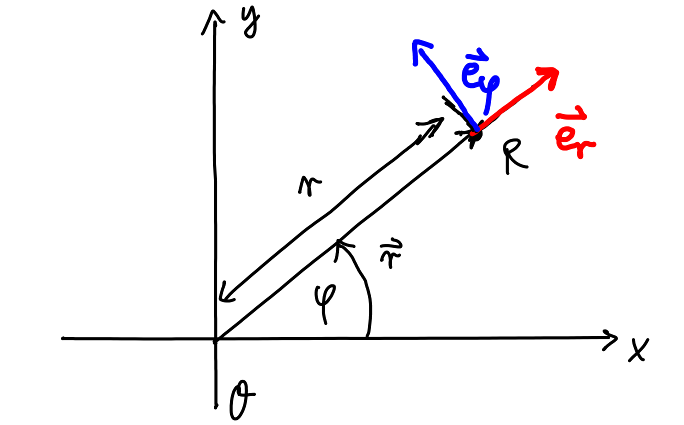
Aufgabe 1: Koordinatensysteme und Analytische Geometrie

(i)

$$\vec{r} = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} = -1 \cdot \vec{e}_x + \sqrt{3} \cdot \vec{e}_y \,, \qquad \vec{e}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{e}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

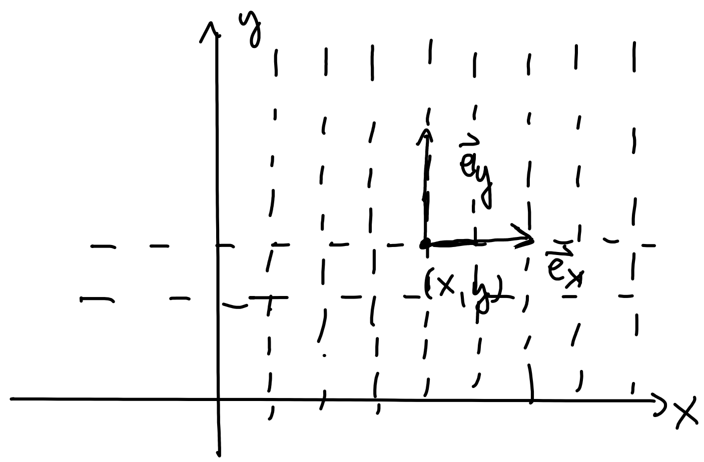
Polarkoordinaten



$$R = (r, \varphi)$$

$$\vec{e}_r = \frac{\vec{r}}{|\vec{r}|} = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}$$

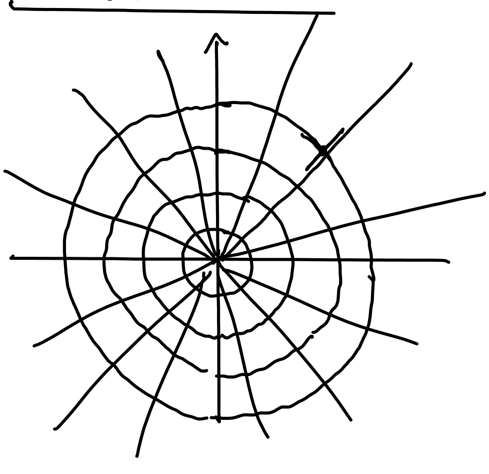
$$\vec{e}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \,, \qquad |\vec{e}_r| = |\vec{e}_\varphi| = 1$$



Zerlegung eines Ortsvektors

$$\vec{r} = r \cdot \vec{e}_r = r \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

Koordinatenlinien



$$\vec{r} = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$$

$$r = \sqrt{\vec{r} \cdot \vec{r}} = \sqrt{1+3} = \sqrt{4} = 2$$

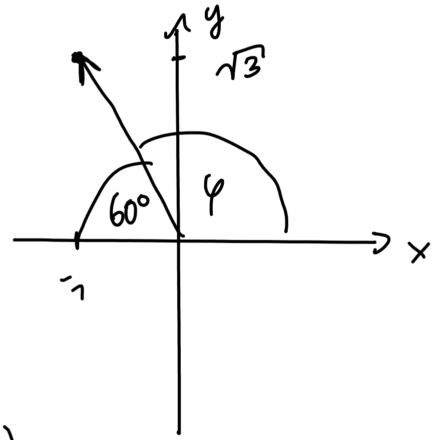
$$x = r \cdot \cos\varphi \Rightarrow \cos\varphi = -\tfrac{1}{2}$$

$$y = r \cdot \sin\varphi \Rightarrow \sin\varphi = \tfrac{\sqrt{3}}{2}$$

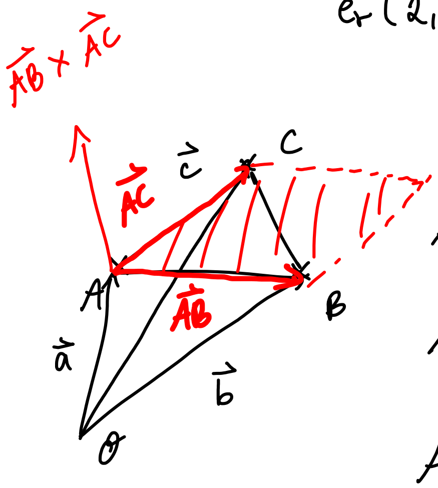
$$\varphi = 120^\circ \qquad (r, \varphi) = (2, 120^\circ) = \left(2, \tfrac{2\pi}{3}\right)$$

$$-180^\circ \le \varphi < 180^\circ \qquad \underline{\underline{0 \le \varphi \le 360^\circ}}$$

$$\vec{r} = \underbrace{2 \begin{pmatrix} -\tfrac{1}{2} \\ \tfrac{\sqrt{3}}{2} \end{pmatrix}}_{\vec{e}_r\left(2, \tfrac{2\pi}{3}\right)}$$



(ii)



$$\vec{AC} = \vec{c} - \vec{a} = \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix}$$

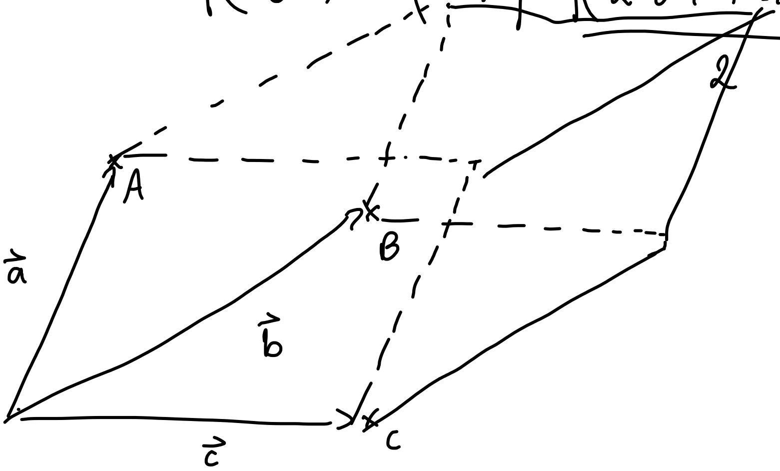
$$\vec{AB} = \vec{b} - \vec{a} = \begin{pmatrix} 2 \\ -10 \\ 0 \end{pmatrix}$$

$$A_{ABC} = \frac{|\vec{AB} \times \vec{AC}|}{2}$$

$$A_{ABC} = \frac{1}{2} \left| \begin{pmatrix} 2 \\ -10 \\ 0 \end{pmatrix} \times \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix} \right| = \left| \begin{pmatrix} 0 \\ 0 \\ 2 \cdot 0 + 7 \cdot 10 \end{pmatrix} \right| = \left| \begin{pmatrix} 0 \\ 0 \\ 70 \end{pmatrix} \right| \frac{1}{2} = 35$$

$$\frac{}{2}$$

(iii)



$$V = |(\vec{a} \times \vec{b}) \cdot \vec{c}| = \left(\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right) \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \cdot 3 - 1 \cdot 0 \\ 1 \cdot 1 - (-1 \cdot 3) \\ -1 \cdot 0 - 2 \cdot 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 4 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 12 + 8 - 2 = 18$$

(iv) a) $g_1: \vec{r} = \begin{pmatrix} x \\ 3x-7 \end{pmatrix} = \begin{pmatrix} 0 \\ -7 \end{pmatrix} + x \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$g_2: \vec{r} = \begin{pmatrix} x \\ 7x-3 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} + x \cdot \begin{pmatrix} 1 \\ 7 \end{pmatrix}$

$$\cos\varphi = \frac{\begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 7 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ 7 \end{pmatrix}\right|} = \frac{1+21}{\sqrt{1+9}\sqrt{1+49}}$$

$$= \frac{22}{\sqrt{10} \cdot \sqrt{50}} = \frac{22}{\sqrt{500}}$$

$\varphi \approx 10{,}3°$

b) $g_1: \vec{r} = \begin{pmatrix} 0 \\ -7 \end{pmatrix} + x \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$g_2: \vec{r} = \begin{pmatrix} 0 \\ 14 \end{pmatrix} + x \cdot \begin{pmatrix} 1 \\ 7 \end{pmatrix}$

Richtungsvektoren gleich zu a) $\Rightarrow$ φ gleich

$\varphi \approx 10{,}3°$

(v) $g_1: \vec{x} = \begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix} + a \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

$g_2: \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} + b \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}$

Schnittpunkt

$$\begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix} + a \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} + b \cdot \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4+a \\ -1-2a \\ 4+3a \end{pmatrix} = \begin{pmatrix} 3b \\ b \\ 6-5b \end{pmatrix} \begin{matrix} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{matrix}$$

II $b = -1 - 2a$

II in I : $4 + a = -3 - 6a \Rightarrow 7a + 7 = 0 \Rightarrow a = -1$

$b = -1 - 2a = -1 + 2 = 1$

a in g_1:

$$\begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$S = (3, 1, 1)$

$$\cos(\varphi) = \frac{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}\right| \left|\begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}\right|} = \frac{3 - 2 - 15}{\sqrt{1+4+9} \cdot \sqrt{9+1+25}}$$

$$= \frac{-14}{\sqrt{14} \cdot \sqrt{35}} = -\sqrt{\frac{14}{35}} = -\sqrt{\frac{2}{5}}$$

$$\varphi \approx 50{,}768° \quad (129{,}232°)$$

(vi)

$E_1: \vec{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -4 \\ 1 \\ -7 \end{pmatrix}$, $E_2: \begin{pmatrix} -4 \\ 3 \\ 2 \end{pmatrix} \vec{x} = 5$

E_1 in Normalenform:

$$\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} -4 \\ 1 \\ -7 \end{pmatrix} = \begin{pmatrix} -21 - 1 \\ -4 - 7 \\ -1 + 12 \end{pmatrix} = \begin{pmatrix} -22 \\ -11 \\ 11 \end{pmatrix}$$

$$\vec{n} = \frac{\begin{pmatrix} -22 \\ -11 \\ 11 \end{pmatrix}}{\left|\begin{pmatrix} -22 \\ -11 \\ 11 \end{pmatrix}\right|} = \frac{11 \cdot \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}}{11 \cdot \left|\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\right|} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{4+1+1}}$$

$$= \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{6}}$$

wähle $\vec{n} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{6}}$

$$\frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \vec{x} = 0$$ Hessesche Normalenform

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} (\vec{x} - \vec{a}) = 0 \, , \quad \vec{a} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \vec{x} - \underbrace{\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \vec{a}} = 0$$

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 2 \cdot 0 + 1 \cdot 1 - 1 \cdot 1 = 0$$

$$\Rightarrow \quad \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \vec{x} = 0$$

$E_1: \quad 2x_1 + x_2 - x_3 = 0 \qquad \text{I}$

$E_2: \quad -4x_1 + 3x_2 + 2x_3 = 5 \qquad \text{II}$

aus I: $x_3 = 2x_1 + x_2$ (*) in II

$$-\cancel{4x_1} + 3x_2 + \cancel{4x_1} + 2x_2 = 5 \Rightarrow 5x_2 = 5 \Rightarrow x_2 = 1$$

$x_2 = 1$ in (*): $\quad x_3 = 2x_1 + 1$

$$\vec{x} = \begin{pmatrix} x_1 \\ 1 \\ 2x_1 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$\rightarrow$ Lösungsmenge : $g: \quad \vec{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

Aufgabe 2: Vektorrechnung

a) $(\vec{a} \times \vec{b}) \perp \vec{a}$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

$$\begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \underline{a_1 a_2 b_3} - \underline{a_1 a_3 b_2} + \underline{a_2 a_3 b_1} - \underline{a_2 a_1 b_3} + \underline{a_3 b_2 a_1} - \underline{a_3 a_2 b_1} = 0$$

b)

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b})$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \left(\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \times \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_2 c_3 - b_3 c_2 \\ b_3 c_1 - b_1 c_3 \\ b_1 c_2 - b_2 c_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_2 b_1 c_2 - a_2 b_2 c_1 - a_3 b_3 c_1 + a_3 b_1 c_3 \\ a_3 b_2 c_3 - a_3 b_3 c_2 - a_1 b_1 c_2 + a_1 b_2 c_1 \\ a_1 b_3 c_1 - a_1 b_1 c_3 - a_2 b_2 c_3 + a_2 b_3 c_2 \end{pmatrix} \quad \leftarrow$$

$\vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b})$:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \cdot (a_1 c_1 + a_2 c_2 + a_3 c_3) - \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} (a_1 b_1 + a_2 b_2 + a_3 b_3)$$

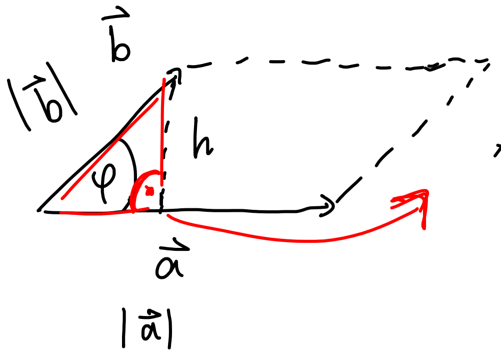
$$= \begin{pmatrix} \underline{a_1 c_1 b_1} + a_2 c_2 b_1 + a_3 c_3 b_1 - \underline{c_1 a_1 b_1} - c_1 a_2 b_2 - c_1 a_3 b_3 \\ b_2 a_1 c_1 + \underline{b_2 a_2 c_2} + b_2 a_3 c_3 - c_2 a_1 b_1 - \underline{c_2 a_2 b_2} - c_2 a_3 b_3 \\ b_3 a_1 c_1 + \underline{b_3 a_2 c_2} + \underline{b_3 a_3 c_3} - c_3 a_1 b_1 - \underline{c_3 a_2 b_2} - \underline{c_3 a_3 b_3} \end{pmatrix}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$$

$$= \underline{\vec{b}(\vec{a}\vec{c})} - \underline{\vec{c}(\vec{a}\vec{b})} + \underline{\vec{c}(\vec{b}\vec{a})} - \underline{\vec{a}(\vec{b}\vec{c})} + \underline{\vec{a}(\vec{c}\vec{b})} - \underline{\vec{b}(\vec{c}\vec{a})} = 0$$

$$\vec{a}\cdot\vec{c} = \vec{c}\cdot\vec{a}$$

c) $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b}) + (\vec{a}\vec{b})(\vec{a}\cdot\vec{b}) = (\vec{a}\vec{a})(\vec{b}\cdot\vec{b})$



$A = |\vec{a}| \cdot h \qquad A = |\vec{a}|\,|\vec{b}| \cdot \sin\varphi$

$h = |\vec{b}| \cdot \sin\varphi$

$$(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b}) = \left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\right)$$

$$\begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

$$= (a_2 b_3 - a_3 b_2)(a_2 b_3 - a_3 b_2)$$
$$+ (a_3 b_1 - a_1 b_3)\cdot(a_3 b_1 - a_1 b_3)$$
$$+ (a_1 b_2 - a_2 b_1)\cdot(a_1 b_2 - a_2 b_1)$$

$$= \underline{a_2^2 b_3^2} - a_2 a_3 b_3 b_2 - a_3 a_2 b_2 b_3 + \underline{a_3^2 b_2^2}$$
$$+ \underline{a_3^2 b_1^2} - \underline{2 a_3 a_1 b_1 b_3} + \underline{a_1^2 b_3^2}$$
$$+ \underline{a_1^2 b_2^2} - 2 a_2 b_1 a_1 b_2 + \underline{a_2^2 b_1^2}$$

$$(\vec{a}\vec{b})(\vec{a}\vec{b}) = (a_1 b_1 + a_2 b_2 + a_3 b_3)\cdot(a_1 b_1 + a_2 b_2 + a_3 b_3)$$

$$= a_1^2 b_1^2 + a_2^2 b_2^2 + a_3^2 b_3^2 + 2 a_1 b_1 a_2 b_2 + 2 a_1 b_1 a_3 b_3$$

$$+ 2 a_2 b_2 a_3 b_3$$

also $$(\vec{a} \times \vec{b})(\vec{a} \times \vec{b}) + (\vec{a}\vec{b})(\vec{a}\vec{b})$$

$$= a_1^2 b_1^2 + a_2^2 b_2^2 + a_3^2 b_3^2 + a_1^2 b_2^2 + a_2^2 b_1^2$$

$$+ a_1^2 b_3^2 + a_3^2 b_1^2 + a_2^2 b_3^2 + a_3^2 b_2^2$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

$$= (\vec{a}\vec{a})(\vec{b}\vec{b})$$

$$A = |\vec{a}||\vec{b}| \sin\varphi$$

$$(\vec{a} \times \vec{b})(\vec{a} \times \vec{b}) = |\vec{a} \times \vec{b}|^2 = A^2 = |\vec{a}|^2 |\vec{b}|^2 \underbrace{\sin^2\varphi}_{= 1 - \cos^2\varphi}$$

Winkelformel: $$\cos^2\varphi = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right)^2$$

$$A^2 = |\vec{a}|^2 |\vec{b}|^2 \cdot \left(1 - \frac{(\vec{a} \cdot \vec{b})^2}{|\vec{a}|^2 |\vec{b}|^2}\right)$$

$$= |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

$$= (\vec{a} \times \vec{b})^2$$

$$\Rightarrow \boxed{(\vec{a} \times \vec{b})(\vec{a} \times \vec{b}) = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2}$$