Vorlesung 7 Lineare Algebra II

Aufgabe 1: Gleichungssysteme

(i)

a)

$$\left.\begin{aligned} 2x + 3y &= 8 \\ 1x - 1y &= -1 \end{aligned}\right\} \longrightarrow \left(\begin{array}{cc|c} 2 & 3 & 8 \\ 1 & -1 & -1 \end{array}\right)$$

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 3 & 8 \\ 1 & -1 & -1 \end{array}\right) \rightsquigarrow \begin{array}{c} \mathrm{I} - 2\mathrm{II} \\ \\ \end{array} \left(\begin{array}{cc|c} 0 & 3+2 & 8+2 \\ 1 & -1 & -1 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 0 & 5 & 10 \\ 1 & -1 & -1 \end{array}\right) \rightsquigarrow \begin{array}{c} \mathrm{I}/5 \\ \\ \end{array} \left(\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & -1 & -1 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} \mathrm{I} \\ \mathrm{II}+\mathrm{I} \end{array} \left(\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 1 \end{array}\right) \longrightarrow \quad x = 1, \quad y = 2$$

b)

$$\left(\begin{array}{cc|c} 1 & -2 & -7 \\ 2 & 3 & 0 \end{array}\right) \rightsquigarrow \begin{array}{c} \mathrm{I} \\ \mathrm{II} - 2\mathrm{I} \end{array} \left(\begin{array}{cc|c} 1 & -2 & -7 \\ 0 & 7 & 14 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} \mathrm{I} \\ \mathrm{II}/7 \end{array} \left(\begin{array}{cc|c} 1 & -2 & -7 \\ 0 & 1 & 2 \end{array}\right) \rightsquigarrow \begin{array}{c} \mathrm{I} + 2\mathrm{II} \\ \\ \end{array} \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array}\right)$$

$$x = -3, \quad y = 2$$

c) $$\left(\begin{array}{ccc|c} 5 & 1 & 2 & 3 \\ -2 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 \end{array}\right) \rightsquigarrow \begin{array}{c} \mathrm{I}-5\mathrm{III} \\ \mathrm{II}+2\mathrm{III} \\ \mathrm{III} \end{array} \left(\begin{array}{ccc|c} 0 & -4 & -3 & 3 \\ 0 & 2 & 3 & -1 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} \mathrm{I}+2\mathrm{II} \\ \mathrm{II} \\ 2\mathrm{III}-\mathrm{II} \end{array} \left(\begin{array}{ccc|c} 0 & 0 & 3 & 1 \\ 0 & 2 & 3 & -1 \\ 2 & 0 & \underline{-1} & 1 \end{array}\right) \rightsquigarrow \begin{array}{c} \mathrm{I} \\ \mathrm{II}-\mathrm{I} \\ 3\mathrm{III}+\mathrm{I} \end{array} \left(\begin{array}{ccc|c} 0 & 0 & 3 & 1 \\ 0 & 2 & 0 & -2 \\ 6 & 0 & 0 & 4 \end{array}\right)$$

$$\begin{array}{c} \mathrm{I}/3 \\ \mathrm{II}/2 \\ \mathrm{III}/6 \end{array} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 1/3 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 2/3 \end{array}\right)$$

$x = 2/3$

$y = -1$

$z = 1/3$

(ii)

a) $$\left.\begin{array}{l} 2x + 3y = b \\ \underline{x} + ay = 4 \end{array}\right\} \left(\begin{array}{cc|c} 2 & 3 & b \\ 1 & a & 4 \end{array}\right)$$

$2x$

$$\left(\begin{array}{cc|c} 2 & 3 & b \\ 1 & a & 4 \end{array}\right) \rightsquigarrow \begin{array}{c} \underline{\mathrm{I}-2\mathrm{II}} \\ \end{array} \left(\begin{array}{cc|c} 0 & 3-\underline{2a} & b-8 \\ 1 & \underline{a} & 4 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} \mathrm{I} \\ 2\mathrm{II} \end{array} \left(\begin{array}{cc|c} 0 & 3-2a & b-8 \\ 2 & 2a & 8 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} \mathrm{I} \\ \mathrm{I}+\mathrm{II} \end{array} \left(\begin{array}{cc|c} 0 & 3-2a & b-8 \\ 2 & 3 & b \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} \mathrm{I} \\ -3\mathrm{I} + (3-2a)\mathrm{II} \end{array} \left(\begin{array}{cc|c} 0 & 3-2a & b-8 \\ (3-2a)2 & 0 & \underbrace{(3-2a)b - 3(b-8)}_{\cancel{3b} - 2ab - \cancel{3b} + 24} \end{array} \right)$$

$$\rightsquigarrow \begin{array}{c} \mathrm{I} \\ \mathrm{II}/2 \end{array} \left(\begin{array}{cc|c} 0 & 3-2a & b-8 \\ 3-2a & 0 & -ab+12 \end{array} \right)$$

0

1. Fall $a = 3/2 \Rightarrow 3-2a = 0$

$$\left(\begin{array}{cc|c} 0 & 0 & b-8 \\ 0 & 0 & -\frac{3}{2}b + 12 \end{array} \right)$$

$\underbrace{}_{A}$ $\underbrace{}_{\vec{b}}$

$A\vec{x} = \vec{b}$

$0 \Rightarrow 0$

① $0 = 0 \cdot x + 0 \cdot y = b - 8$

② $0 = 0 \cdot x + 0 \cdot y = -3/2\, b + 12$

① $\Rightarrow b = 8$ in ② $\quad -\frac{3}{2} \cdot 8 + 12 = -\frac{24}{2} + 12 = 0$

$\longrightarrow \quad a = 3/2, \quad b = 8$

2. Fall: $a \neq 3/2 \Rightarrow 3-2a \neq 0$

$$\left(\begin{array}{cc|c} 0 & 3-2a & b-8 \\ 3-2a & 0 & -ab+12 \end{array} \right)$$

$$\rightsquigarrow \begin{array}{c} \mathrm{I}/(3-2a) \\ \mathrm{II}/(3-2a) \end{array} \left(\begin{array}{cc|c} 0 & 1 & \frac{b-8}{3-2a} \\ 1 & 0 & \frac{12-ab}{3-2a} \end{array} \right)$$

$$x = \frac{12-ab}{3-2a} \quad \wedge \quad y = \frac{b-8}{3-2a}$$

$$\frac{y}{x} = \frac{b-8}{\cancel{3-2a}} \cdot \frac{\cancel{3-2a}}{12-ab} = \frac{b-8}{12-ab}$$

$$\rightarrow \quad y = \frac{b-8}{12-ab}\, x$$

b)

$$\left.\begin{aligned} x + 2y - z &= s \\ 0\cdot z + x + y &= 1 \\ 0\cdot x + y - z &= 2 \end{aligned}\right\} \left(\begin{array}{ccc|c} 1 & 2 & -1 & s \\ 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

$$\overset{\mathrm{I}-\mathrm{II}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 0 & 1 & -1 & s-1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right) \rightsquigarrow \overset{\mathrm{I}-\mathrm{III}}{} \left(\begin{array}{ccc|c} 0 & 0 & 0 & s-3 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

2 Zeilen gleich

$\rightarrow$ nicht invertierbar

$$0\cdot x + 0\cdot y + 0\cdot z = s-3 \Rightarrow s = 3$$

(2) $x + y = 1 \Rightarrow y = 1 - x$

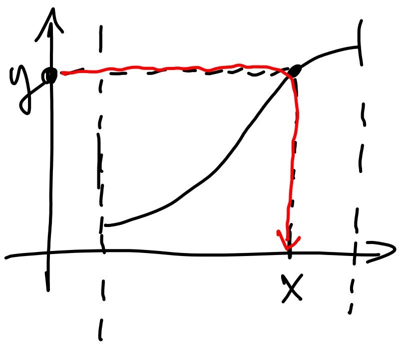
(3) $y - z = 2 \Rightarrow z = y - 2 = 1 - x - 2 = -1 - x$

$$\vec{r} = \begin{pmatrix} x \\ 1-x \\ -1-x \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + x \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad \text{Gerade}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \vec{x} = \begin{pmatrix} s \\ 1 \\ 2 \end{pmatrix}$$

1. es existiert nur dann ein lösendes $\vec{x}$, falls $s = 3$

dann liegt x auf einer Geraden



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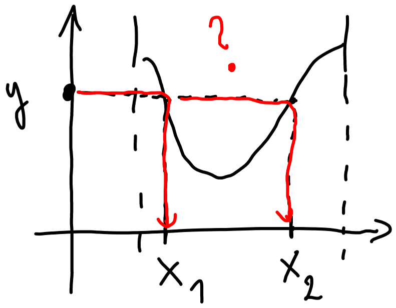

2. Matrizenrechnung

(i)

Modell \ Bestandteil	Schubladen	Böden	Türen
	1	2	3
X	6	12	2
Y	4	12	3
Z	6	14	4

Produktion: $x = 15, \ y = 9, \ z = 6$ $\quad \vec{p} = \begin{pmatrix} 15 \\ 9 \\ 6 \end{pmatrix}$

Produktdimension

$$f(\vec{p}) : P \longrightarrow B$$

$$\vec{p} \longrightarrow \vec{b} = V \cdot \vec{p}$$

$$\vec{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} s \\ b \\ t \end{pmatrix}$$

$$V \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ b \\ t \end{pmatrix}$$

$$\vec{p} \in \mathbb{N}^{3 \times 1}$$

$$V = \begin{array}{ccc|c} & & & \\ \hline \end{array}$$

$$V = \begin{array}{c c} \begin{pmatrix} 6 & 4 & 6 \\ 12 & 12 & 14 \\ 2 & 3 & 4 \end{pmatrix} & \begin{array}{c} s \\ b \\ t \end{array} \\ \begin{array}{ccc} x & y & z \end{array} & \end{array}$$

$$V\vec{p} = \begin{pmatrix} 6 & 4 & 6 \\ 12 & 12 & 14 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 15 \\ 9 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \cdot 15 + 4 \cdot 9 + 6 \cdot 6 \\ 12 \cdot 15 + 12 \cdot 9 + 14 \cdot 6 \\ 2 \cdot 15 + 3 \cdot 9 + 4 \cdot 6 \end{pmatrix}$$

$$= \begin{pmatrix} 162 \\ 372 \\ 87 \end{pmatrix}$$

(ii) <u>Matrixprodukte</u>

a)

$$\begin{array}{c} z \end{array} \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 5 & -1 \end{pmatrix} \begin{array}{c} s \\ \begin{pmatrix} 2 & 1 & 1 \\ 4 & -3 & -2 \end{pmatrix} \end{array} = \begin{array}{c} z \end{array} \begin{array}{c} s \\ \begin{pmatrix} 6 & -2 & -1 \\ 8 & -6 & -4 \\ 6 & 8 & 7 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & -3 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 2+5 & 2+2-1 \\ 4-10 & 4-6+2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 3 \\ -6 & 0 \end{pmatrix}$$

b)

$$\begin{pmatrix} -2 & 3 & 1 \\ 1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} -3 & 5 & -2 \\ 1 & 2 & -3 \\ 2 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} \overbrace{6+3+2}^{11} & \overbrace{-10+6+4}^{0} & \overbrace{4-9+5}^{0} \\ \overbrace{-3+1+2}^{0} & \overbrace{5+2+4}^{11} & \overbrace{-2-3+5}^{0} \\ \underbrace{-2+2}_{0} & \underbrace{-4+4}_{0} & \underbrace{6+5}_{11} \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix} = 11 \cdot \mathbb{1}$$

(iii)

a)

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & -3 & 0 & 1 \end{array}\right) \rightsquigarrow \begin{array}{c} \mathrm{I} \\ \mathrm{II}-2\mathrm{I} \end{array} \left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & -9 & -2 & 1 \end{array}\right)$$

$$\rightsquigarrow \quad 3\mathrm{I}+\mathrm{II} \left(\begin{array}{cc|cc} 6 & 0 & 1 & 1 \\ 0 & -9 & -2 & 1 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} \mathrm{I}/6 \\ \mathrm{II}/(-9) \end{array} \left(\begin{array}{cc|cc} 1 & 0 & 1/6 & 1/6 \\ 0 & 1 & \frac{2}{9} & -1/9 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1/6 & 1/6 \\ 2/9 & -1/9 \end{pmatrix}$$

Probe:

$$\begin{pmatrix} 2 & 3 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 1/6 & 1/6 \\ 2/9 & -1/9 \end{pmatrix} = \begin{pmatrix} 2/6 + 6/9 & 2/6 - 3/9 \\ 4/6 - 6/9 & 4/6 + 3/9 \end{pmatrix}$$

$$= \begin{pmatrix} \overbrace{1/3 + 2/3}^{1} & \cancel{1/3} - \cancel{1/3} \\ \cancel{2/3} - \cancel{2/3} & \underbrace{2/3 + 1/3}_{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

b)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \begin{array}{c} \mathrm{I} \\ \mathrm{II}-2\mathrm{I} \\ \mathrm{III}-3\mathrm{I} \end{array} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} \mathrm{I}+\mathrm{III} \\ \mathrm{II} \\ \mathrm{III}-3\mathrm{II} \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & 3 & -3 & 1 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} I + III \\ -II \\ -III \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -3 & 3 & -1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \middle| \underbrace{\begin{array}{ccc} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{array}}_{A^{-1}} \right)$$

c) $$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} I \\ II - 2I \\ III + I \end{array} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 4 & 0 & -2 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} 2I + II - III \\ \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & -1 & 1 & -1 \\ 0 & 4 & 0 & -2 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} 2I \\ 1 \cdot II \\ 2III \end{array} \left(\begin{array}{ccc} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{array} \middle| \underbrace{\begin{array}{ccc} -2 & 2 & -2 \\ -2 & 1 & 0 \\ 2 & 0 & 2 \end{array}}_{4A^{-1}} \right)$$

$$A^{-1} = \frac{1}{4} \begin{pmatrix} -2 & 2 & -2 \\ -2 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

d) $$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 5 & 0 & 1 & 0 \\ -1 & 1 & -5 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III}-\mathrm{I}+\mathrm{II} \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right)$$

$\rightarrow$ nicht invertierbar!

(iv) $A\vec{x} = \vec{b}$

d) $$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 2 & 0 & 5 & 1 \\ -1 & 1 & -5 & -2 \end{array}\right)$$

$$\vec{b} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

nichts auf den

$$\rightsquigarrow \begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III}-\mathrm{I}+\mathrm{II} \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 2 & 0 & 5 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right) \quad 3$$

$$0 \cdot x + 0 \cdot y + 0 \cdot z = -4$$

$$\Rightarrow 0 = -4$$

Gerade auf den

$$\vec{b} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 2 & 0 & 5 & 1 \\ -1 & 1 & -5 & 2 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III}-\mathrm{I}+\mathrm{II} \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 2 & 0 & 5 & 1 \\ 0 & 0 & 0 & \underbrace{2-3+1}_{0} \end{array}\right)$$

$\hookrightarrow$ (1) $x + y = 3 \rightarrow x = 3 - y$

(2) $2x + 5z = 1 \rightarrow 6 - 2y + 5z = 1 \rightarrow 5z = -5 + 2y$

(3) $0x + 0y + 0z = 0$ ✓ $\quad z = -1 + \frac{2}{5}y$

$$\vec{x} = \begin{pmatrix} 3 - y \\ y \\ -1 + \frac{2}{5}y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + y \cdot \begin{pmatrix} -1 \\ 1 \\ 2/5 \end{pmatrix}$$

$\rightarrow$ Gerade