Vorrechnen 10

Aufgabe 1 : Differenzialrechnung

a) $$x^x = e^{\ln(x^x)} = e^{x\ln(x)} \overset{KR, PR}{=} \underbrace{\overbrace{e^{x\ln(x)}}^{x^x}}_{\left.\frac{d}{dz}e^z\right|_{x\ln(x)=z}} \cdot \Big(\ln(x) + \underbrace{\tfrac{1}{x}\cdot x}_{1}\Big)$$

$\uparrow$ e^z

$$= x^x(\ln(x)+1)$$

b) $$\frac{d}{dx}\left(\sqrt{a}-\sqrt{bx+c}\right)^2 \overset{KR}{=} \cancel{2}\left(\sqrt{a}-\sqrt{bx+c}\right)\cdot\frac{-\frac{1}{\cancel{2}}}{\sqrt{bx+c}}\cdot b$$

$$= -\frac{\sqrt{a}-\sqrt{bx+c}}{\sqrt{bx+c}}\cdot b = -\frac{\sqrt{a}\cdot b}{\sqrt{bx+c}} + b$$

c) $$\frac{(x+1)\sin(x+1)}{(x-1)^2} \overset{QR}{=}$$

$$\frac{\left\{\frac{d}{dx}\left[(x+1)\sin(x+1)\right]\right\}\cdot(x-1)^2 - \left\{\frac{d}{dx}(x-1)^2\right\}\cdot(x+1)\sin(x+1)}{(x-1)^4}$$

$$= \frac{\left[\sin(x+1) + (x+1)\cos(x+1)\right](x-1)^2 - 2(x+1)(x-1)\sin(x+1)}{(x-1)^4}$$

$$= \frac{\sin(x+1) + (x+1)\cos(x+1)}{(x-1)^2} - \frac{2(x+1)\sin(x+1)}{(x-1)^3}$$

d) $$\ln\left(\sqrt{x^2+\sin^2(x)}\right) \overset{KR\times 2}{=} \underbrace{\frac{1}{\sqrt{x^2+\sin^2(x)}}}_{\left.\frac{d}{dz}\ln(z)\right|_{z=\sqrt{x^2+\sin^2(x)}}} \cdot \underbrace{\frac{1}{\cancel{2}\sqrt{x^2+\sin^2(x)}}}_{\left.\frac{d}{dy}\sqrt{y}\right|_{y=x^2+\sin^2(x)}} \cdot \left(\cancel{2}x + \cancel{2}\sin x\cos x\right)$$

$$= \frac{1}{|x^2+\sin^2(x)|}\cdot(x+\sin x\cos x) = \frac{x+\sin x\cos x}{x^2+\sin^2(x)}$$

$= x^2+\sin^2(x) \geq 0$

Aufgabe 2 Integralrechnung

(i) a) $\int x^2\,dx = \frac{1}{3}x^3 + c$

b) $\int x^n\,dx \quad , n \neq -1 \quad = \frac{1}{n+1}x^{n+1} + c$

c) $\int \frac{1}{x}\,dx = \ln(|x|) + c$

d) $\int \frac{1}{x+23}\,dx = \ln(|x+23|) + c$

KR rückwärts

$$\frac{1}{x+23}$$

e) $\int \frac{-5}{x^6}\,dx = x^{-5} + c$

Potenzregel rückwärts

f) $\int e^{-5x}\,dx = -\frac{1}{5}e^{-5x} + c$

KRR

g) $\int \sin x \cos x \,dx = \frac{1}{2}\sin^2 x + c$

KRR

1. Methode: KR rückwärts $\quad f(z) = z^2 \quad g(x) = \sin(x) = z$

$$f(g(x)) = \sin^2 x$$

$$\frac{d}{dx} f(g(x)) = 2\sin x \cos x$$

2. Methode: partielle Integration

$$\underbrace{\int \underbrace{\sin x}_{f(x)} \underbrace{\cos x}_{g(x)}\,dx}_{I} = \sin x \sin x - \underbrace{\int \cos x \sin x \,dx}_{-I} \quad \Big| + I$$

$$= f(x)\,G(x) - \int f'(x)\,G(x)\,dx$$

$2 \cdot \int \sin x \cos x \,dx = \sin x \sin x$

$\Rightarrow \int \sin x \cos x \,dx = \frac{1}{2}\sin^2 x + c$

3. Methode : Substitution

$$\boxed{dx = \frac{dx}{dy}\, dy = \frac{1}{y'}\, dy}$$

$$\int \sin x \cos x \, dx \rightarrow y = \sin x, \quad y(x) \quad \frac{dy}{dx} = \cos x$$

$$= \int \sin x \cos x \underbrace{\frac{dx}{dy} \cdot dy}_{} = \int \sin x \cos x \underbrace{\frac{dy}{dy}}_{=1} dx \qquad \frac{dx}{dy} = \frac{1}{\cos x}$$

$$= \int \underbrace{\sin x}_{y} \cancel{\cos x} \frac{1}{\cancel{\cos x}}\, dy \qquad dx \mathrel{\hat{=}} \lim_{x \to x_0} \underbrace{(x - x_0)}_{\Delta x}$$

$$= \int y \, dy = \frac{1}{2} y^2 = \frac{1}{2} (\sin x)^2 \qquad DQ: f' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

h) $I = \int (x \cos x + \sin x)\, dx = \underbrace{\int x \cos x \, dx}_{?} + \underbrace{\int \sin x \, dx}_{-\cos x + c} \qquad = \frac{dy}{dx}$

$$\int \underset{f(x)}{x}\, \underset{g(x)}{\cos x}\, dx \overset{P.I}{=} x \cdot \sin x - \underbrace{\int 1 \cdot \sin x \, dx}_{-\cos x} = x \cdot \sin x + \cos x$$

$$I = x \cdot \sin x + \cancel{\cos x} - \cancel{\cos x} + c = x \cdot \sin x + c$$

PR $\leftrightarrow$ partielle Integration

KR $\leftrightarrow$ Substitution

(i)
$$\int \frac{x}{1+x^2}\, dx = \cancel{x \cdot (1+x^2)^{-1} = \frac{x}{1+x^2}}$$

$$= \frac{1}{2} \ln(1+x^2) + c$$

$$\frac{1}{\cancel{2}} \cdot \frac{1}{1+x^2} \cdot \cancel{2}x = \frac{x}{1+x^2}$$

Substitution : $y = 1 + x^2 \qquad \frac{dy}{dx} = 2x$

$$\int \frac{x}{1+x^2}\, dx = \int \frac{x}{1+x^2} \frac{dx}{dy}\, dy = \frac{1}{2} \int \frac{\cancel{x}}{\underset{y}{1+x^2}} \frac{1}{2\cancel{x}}\, dy \qquad \frac{dx}{dy} = \frac{1}{2x}$$

$$= \frac{1}{2} \int \frac{1}{y}\, dy = \frac{1}{2} \ln(y) = \frac{1}{2} \ln(1+x^2) = \ln\left(\sqrt{1+x^2}\right)$$

3. Methode: $\int \frac{x}{1+x^2}\, dx = \int \frac{x}{(x-i)(x+i)}\, dx$

$$x^2+1=0 \Rightarrow x^2 = -1 \Rightarrow x = \pm \underbrace{\sqrt{-1}}_{i}$$

$$\frac{x}{(x-i)(x+i)} \overset{!}{=} \frac{A}{x-i} + \frac{B}{x+i} \Rightarrow \frac{(A+B)x + (A-B)i}{(x+i)(x-i)} \overset{!}{=} x$$

$A = B$, $A + B = 1$

$2A = 1 \Rightarrow A = B = \frac{1}{2}$

$$\int \frac{1/2}{x-i}\, dx + \int \frac{1/2}{x+i}\, dx$$

$$= \frac{1}{2} \ln(x-i) + \frac{1}{2} \ln(x+i) = \frac{1}{2} \ln\left(\underbrace{(x-i)(x+i)}\right)$$

$$x^2 - i^2 = x^2 - \underbrace{\sqrt{-1}}_{-1}{}^2$$

$$= x^2+1$$

$$\Rightarrow \frac{1}{2} \ln(x^2+1)$$

j) $\int \left(6 \sin(4-3x) + 3e^{-2x} + 5\right) dx$

$$= 6 \cdot \int \sin(4-3x)\, dx + 3 \cdot \int e^{-2x}\, dx + 5 \cdot \int dx$$

$$= 6 \cdot \cos(4-3x) \cdot \frac{1}{3} + \left(-\frac{3}{2}\right) e^{-2x} + 5x + c$$

$$= 2 \cos(4-3x) - \frac{3}{2} e^{-2x} + 5x + c$$

k) $\int \frac{x^3 + 2x^2 - 5x - 6}{x^2 - x - 2}\, dx$

Grad Zähler > Grad Nenner → PD

$$(x^3 + 2x^2 - 5x - 6) : (x^2 - x - 2) = x + 3$$
$$-(x^3 - x^2 - 2x)$$
$$3x^2 - 3x - 6$$
$$-(3x^2 - 3x - 6)$$
$$0$$

$$\int (x+3)\,dx = \frac{1}{2}x^2 + 3x + c$$

l) $$\int \frac{x+29}{x^2+3x-28}\,dx$$

Grad Nenner > Grad Zähler $\rightarrow$ PBZ

Nullstellen $\rightarrow$ Linearfaktoren

$$x^2+3x-28=0 \Rightarrow x = -\frac{3}{2} \pm \sqrt{\frac{9}{4}+28}$$

$$= -\frac{3}{2} \pm \sqrt{\frac{9}{4}+\frac{112}{4}}$$

$$= -\frac{3}{2} \pm \sqrt{\frac{121}{4}}$$

$$= -\frac{3}{2} \pm \frac{11}{2} = \frac{8}{2} \vee -\frac{14}{2}$$

$$4 \vee -7$$

$$x^2+3x-28 = (x+7)(x-4)$$

$$(x+29) = (x-4)A + (x+7)B$$

Einsetzen von Nullstellen: $x=4 \rightarrow \underbrace{4+29}_{33} = \underbrace{(4+7)}_{11}B \Rightarrow B=3$

$x=-7 \rightarrow \underbrace{-7+29}_{22} = \underbrace{(-7-4)}_{-11}A \Rightarrow A=-2$

$$\int \frac{x+29}{x^2+3x-28}\,dx = \int\left(\frac{-2}{x+7} + \frac{3}{x-4}\right)dx$$

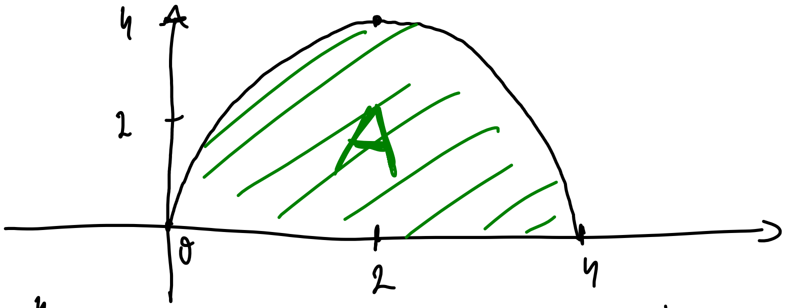
$$= \underbrace{-2\int \frac{1}{x+7}\,dx}_{\ln(|x+7|)+A} + \underbrace{3\int \frac{1}{x-4}\,dx}_{\ln(|x-4|)+B}$$

$$= \ln\left(\frac{|x-4|^3}{(x+7)^2}\right) + \underbrace{B+A}_{c}$$

(11) a) $f(x) = 4x - x^2$ Nullstellen: $x(4-x)=0 \Rightarrow x=0 \vee \underbrace{4-x=0}_{x=4}$

$f'(x) = 4-2x \stackrel{!}{=} 0 \Rightarrow x = 2$

$f(2) = 8 - 4 = 4$



$$\int_0^4 (4x - x^2)\,dx = 2x^2 - \frac{1}{3}x^3 \Big|_0^4 = 2\cdot 16 - \frac{1}{3}\underbrace{4^3}_{64} - 0$$

$$= 32 - \underbrace{\frac{64}{3}}$$

$$= 10.\overline{6} = \frac{32}{3}$$

b) $f(x) = x^3 - 4x^2 + 4x$

Nullstellen: $x = 0$, $x = 2$ ← doppelt

$$x(x^2 - 4x + 4) = 0 \Rightarrow x = 0 \lor \underbrace{x^2 - 4x + 4 = 0}_{(x-2)^2}$$

$$f(x) = x^1 \cdot (x-2)^2$$

$$f'(x) = 3x^2 - 8x + 4 \overset{!}{=} 0$$

$$\Leftrightarrow x^2 - \frac{8}{3}x + \frac{4}{3} = 0 \Rightarrow x = \frac{4}{3} \pm \sqrt{\frac{16}{9} - \frac{4}{3}}$$

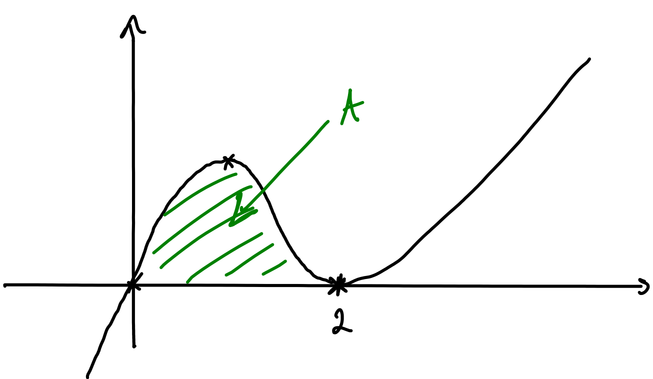
$$= \frac{4}{3} \pm \sqrt{\frac{16}{9} - \frac{12}{9}}$$

$$= \frac{4}{3} \pm \sqrt{\underbrace{\cancel{\tfrac{16}{9} - \tfrac{12}{9}}}_{\frac{4}{9}}}$$

$$= \frac{4}{3} \pm \frac{2}{3} = \underbrace{\frac{6}{3}}_{2} \lor \frac{2}{3}$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

$$\lim_{x \to +\infty} f(x) = +\infty$$



$$\int_0^2 (x^3 - 4x^2 + 4x)\,dx$$

$$= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2\right]_0^2$$

$$= \cancel{} \quad \frac{4}{3}$$

(iii) a) $\displaystyle\int \frac{\cos(x)}{\underbrace{\sqrt{\sin(x)+3}}} \, dx = I$

$y = \sqrt{\sin x + 3}$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{\sin x + 3}} \cdot \cos x$

$$= \int \frac{\cos(x)}{\sqrt{\sin x + 3}} \cdot \underbrace{\frac{dx}{dy}} \cdot dy = \int \frac{\cancel{2\sqrt{\sin x + 3}}}{\cancel{\sqrt{\sin x + 3}}} \frac{\cancel{\cos x}}{\cancel{\cos x}} \, dy$$

$$\left(\frac{dy}{dx}\right)^{-1} = \frac{2\sqrt{\sin x + 3}}{\cos x}$$

$$= 2 \cdot \int dy = 2 \cdot y$$

Rücksubstitution : $y = \sqrt{\sin x + 3} \rightarrow I = 2 \cdot \sqrt{\sin x + 3}$

2. Methode : $y = \sin(x)$ $\qquad \dfrac{dy}{dx} = \cos x$

$$\int \frac{\cos x}{\sqrt{\sin x + 3}} \underbrace{\frac{dx}{dy}}_{\frac{1}{\cos x}} dy = \int \frac{1}{\sqrt{\underbrace{\sin x}_{y} + 3}} \, dy$$

$$= \int \frac{1}{\sqrt{y+3}} \, dy = 2\sqrt{y+3}$$

$$= 2\sqrt{\sin x + 3}$$

b) $$\int \frac{(\ln(x^3))^2}{x} \, dx = \int \frac{(\ln(x) \cdot 3)^2}{x} \, dx = 9 \cdot \int \frac{\ln^2(x)}{x} \, dx$$

$y = \ln(x)$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{x} \Rightarrow \dfrac{dx}{dy} = x$

$$\rightarrow 9 \cdot \int \ln^2(x) \frac{x}{x} \, dy = 9 \cdot \int y^2 \, dy = 9 \cdot \frac{1}{3} y^3 = 3 y^3$$

Rücksubstitution : $3 \ln(x)^3$

c) $$I = \int \frac{e^{2x} + e^x}{e^{2x} - e^x - 2} \, dx \quad , \quad x > \ln(2)$$

$y = e^x$ $\qquad \dfrac{dy}{dx} = e^x = y \Rightarrow \dfrac{dx}{dy} = \dfrac{1}{y}$

$$I = \int \frac{y^2 + y}{y^2 - y - 2} \frac{1}{y} \, dy = \int \frac{y+1}{y^2 - y - 2} \, dy$$

PBZ: $y^2 - y - 2 = 0 \Rightarrow y = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2}$

$= \frac{1}{2} \pm \sqrt{\frac{9}{4}} = \frac{1}{2} \pm \frac{3}{2} = 2 \vee -1$

$y^2 - y - 2 = (y-2)(y+1)$

$$I = \int \frac{\cancel{y+1}}{(y-2)\cancel{(y+1)}} \, dy$$

$y \neq -1 \to$ sonst ja auch kein wohl definiertes Integral

$x > \ln(2) \Rightarrow e^x > 2 \Rightarrow y > 2$

$$\Rightarrow \int \frac{1}{y-2} \, dy = \ln(|y-2|) \overset{RS}{=} \ln(\underbrace{|e^x - 2|}_{>0,\ \text{da } x > \ln(2)}) = \ln(e^x - 2)$$

alternativ:

$$\frac{e^{2x} + e^x}{e^{2x} - e^x - 2} = \frac{e^x \cancel{(e^x+1)}}{\cancel{(e^x+1)}(e^x - 2)} \to \underbrace{\int \frac{e^x}{e^x - 2} \, dx}_{\int \frac{f'(x)}{f(x)} dx}$$

$= \ln(f(x))$ für

$f(x) = e^x - 2$

(iv) a) $\int \underset{\substack{\uparrow \\ f}}{x} \underset{\substack{\uparrow \\ g}}{\sin x} \, dx = -x \cdot \cos x + \underbrace{\int \cos x \, dx}_{\sin x + c}$

$= -x \cos x + \sin x + c$

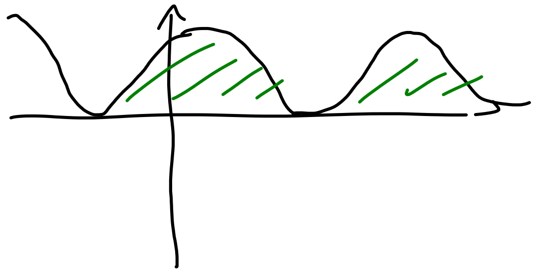
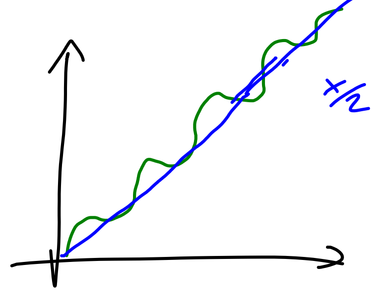
b) $\underbrace{\int \sin^2 x \, dx}_{I} = \int \underbrace{\sin x}_{f(x)} \cdot \underbrace{\sin x}_{g(x)} \, dx$

$= -\sin(x) \cdot \cos(x) + \int \underbrace{\cos x \cos x}_{\cos^2 x = 1 - \sin^2 x} \, dx$

$= -\sin x \cdot \cos x + \underbrace{\int 1 \, dx}_{x} - \underbrace{\int \sin^2 x \, dx}_{I} \quad \Big| + I$

$2 \int \sin^2 x \, dx = -\sin x \cos x + x \Rightarrow \int \sin^2 x \, dx = \frac{-\sin x \cos x}{2} + \frac{x}{2}$

Schwachsinn? linear

c) $\int \ln(x)\, dx = \int \underbrace{\ln(x)}_{f(x)} \cdot \underbrace{1}_{g(x)} \, dx$

$$= \ln(x) \cdot x - \int \underbrace{\frac{1}{\cancel{x}}}_{= \frac{d}{dx}\ln(x)} \cdot \underbrace{\cancel{x}}_{= \int 1\, dx} \, dx = \ln(x) \cdot x - x + c$$

Probe: $\frac{d}{dx}\left(\ln(x) \cdot x - x\right) = \cancel{\frac{1}{x} \cdot x} + \ln(x) \cancel{- 1} = \ln(x)$