Nachtrag / Wiederholung

Blatt 9 Aufgabe 2 Kurvendiskussion

(ii) $f(x) = x^4 - 8x^2 + 9$

Extrema: $f'(x) = 0 \Rightarrow 4x^3 - 16x = 0$

$$\Rightarrow \underbrace{x}_{0} \cdot \underbrace{(4x^2 - 16)}_{0} = 0$$

$x_1 = 0$, $4x^2 - 16 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$\Rightarrow x_2 = -2$, $x_3 = 2$

Monotonie

Bereich	f'	f	
$x < -2$	negativ	streng monoton fallend	} Minimum
$-2 < x < 0$	positiv	streng monoton steigend	} Maximum
$0 < x < 2$	negativ	—"— fallend	} Minimum
$x > 2$	positiv	—"— steigend	

Extrema: schon bekannt durch VZW

alternativ: $f''(x) = 12x^2 - 16$ $\quad f''(0) < 0 \rightarrow$ Maximum

$f''(-2) = f''(2) > 0 \rightarrow$ Minima

$f''(x) = 0 \Rightarrow 12x^2 - 16 = 0 \Rightarrow x^2 = \frac{16}{12} = \frac{4}{3}$

$\Rightarrow x = \pm \frac{2}{\sqrt{3}} \rightarrow$ Wendepunkte

<u>Krümmungsverhalten</u>

Bereich	f''	f
$x < -\frac{2}{\sqrt{3}}$	> 0	Linkskrümmung
$-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$	< 0	Rechtskrümmung
$x > \frac{2}{\sqrt{3}}$	> 0	Linkskrümmung

Nullstellen von $f(x)$: $x^4 - 8x^2 + 9 = 0$ $\quad y = x^2$

$\rightarrow y^2 - 8y + 9 = 0 \Rightarrow y = 4 \pm \sqrt{16 - 9} = 4 \pm \sqrt{7}$

$y = 4 \pm \sqrt{7}$ $\qquad x = \pm\sqrt{4 \pm \sqrt{7}}$

$\Leftrightarrow \quad x^2 = 4 \pm \sqrt{7}$

$\longrightarrow \quad x \in \{ -\sqrt{4+\sqrt{7}}, \; -\sqrt{4-\sqrt{7}}, \; \sqrt{4-\sqrt{7}}, \; \sqrt{4+\sqrt{7}} \}$

$$\sqrt{7} = \sqrt{\frac{36\cdot 7}{36}} = \sqrt{\frac{252}{36}} \approx \sqrt{\frac{256}{36}} = \frac{16}{6} = \frac{8}{3}$$

$$\sqrt{4-\sqrt{7}} \approx \sqrt{4-\frac{8}{3}} = \sqrt{\frac{12-8}{3}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = 1{,}1547\ldots$$

(genauer: $\sqrt{4-\sqrt{7}} = 1{,}1637\ldots$)

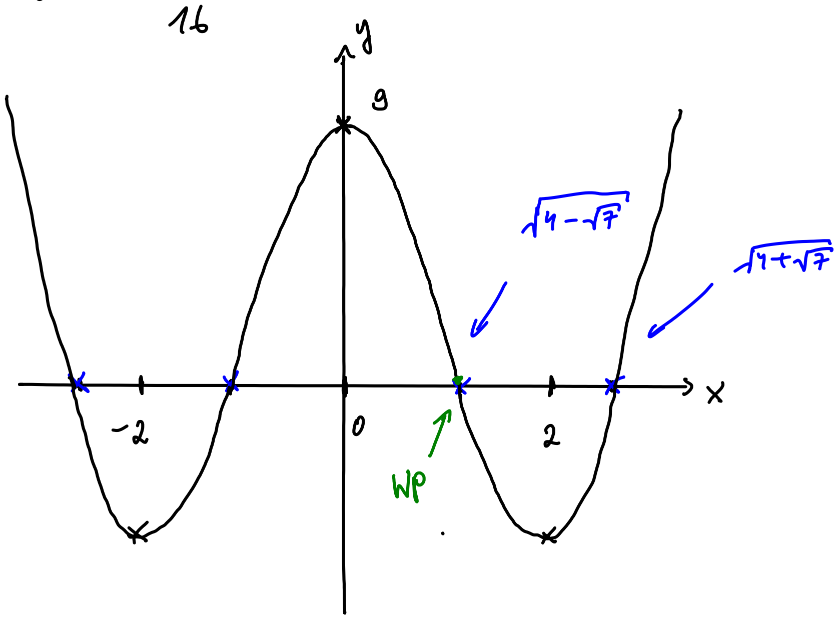
$$\sqrt{4+\sqrt{7}} \approx \sqrt{\frac{12+8}{3}} = \sqrt{\frac{20}{3}} \approx \sqrt{\frac{21}{3}} = \sqrt{7} \approx \frac{8}{3} = 2{,}\overline{6}$$

(genauer: $\sqrt{4+\sqrt{7}} = 2{,}577\ldots.$)

Extrempunkte $\qquad f(x) = x^4 - 8x^2 + 9$

$f(0) = 9$

$f(-2) = f(2) = \underbrace{2^4}_{16} - 8\cdot 4 + 9 = 16 - 32 + 9 = -16 + 9 = -7$



(iii) Krümmung von Funktionen

a) $x^3 - 9x^2 + 3 = f(x)$ $\qquad$ Nullstellen $f''(x)$: $6x - 18 = 0$

$\Rightarrow x = 3$

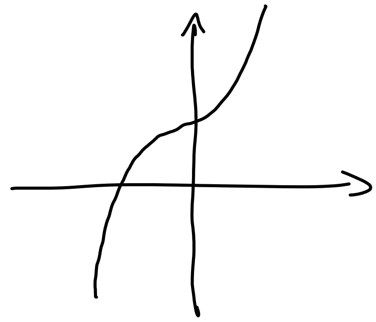
$f'(x) = 3x^2 - 18x$

$f''(x) = 6x - 18$

$f(x)$

$f''(x) > 0$ für $x > 3$ Linkskrümmung

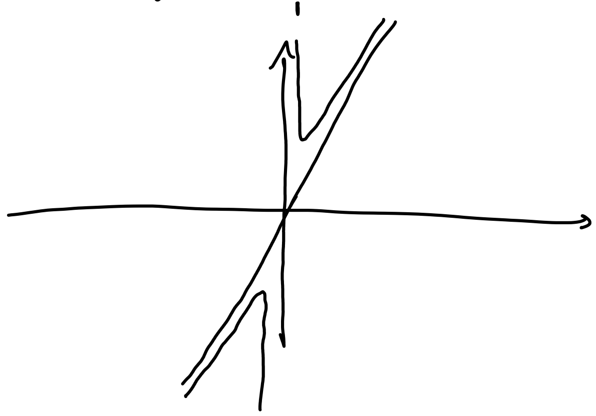
$f''(x) < 0$ für $x < 3$ Rechtskrümmung

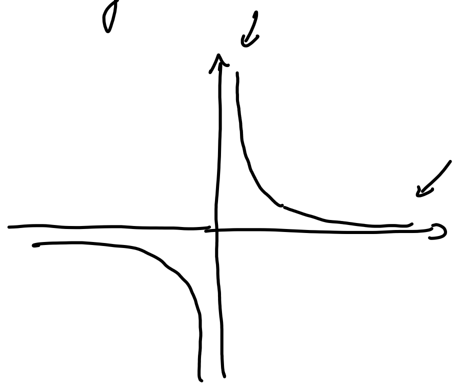


b) $2x + \frac{1}{x} = f(x)$

$f'(x) = 2 - \frac{1}{x^2}$

$f''(x) = \frac{2}{x^3}$ $\begin{cases} > 0, & x > 0 \longrightarrow \text{Linkskrümmung} \\ < 0, & x < 0 \longrightarrow \text{Rechtskrümmung} \end{cases}$

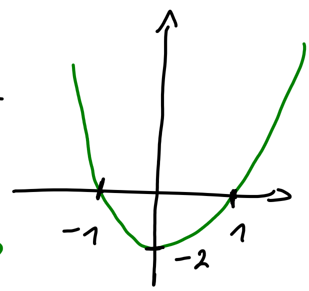




c) $f(x) = e^{-x^2}$ $\quad f'(x) = e^{-x^2} \cdot (-2x)$

$f''(x) = -2e^{-x^2} + e^{-x^2}(-2x)^2$

$= \underbrace{e^{-x^2}}_{>0} \cdot \underbrace{(4x^2 - 2)}$



$f''(x) = 0 \Rightarrow 4x^2 - 2 = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$

$f''(x) > 0, \quad x$ in $(-\infty, -1) \longrightarrow$ $f(x)$ Linkskrümmung

$f''(x) < 0, \quad x$ in $(-1, 1) \longrightarrow$ Rechtskrümmung

$f''(x) > 0, \quad x$ in $(1, +\infty) \longrightarrow$ Linkskrümmung

d) $f(x) = x \cdot e^{-x}$ $\quad f'(x) \overset{PR}{=} e^{-x} + e^{-x} \cdot (-1) \cdot x$

$= e^{-x} \cdot (1 - x)$

$f''(x) = e^{-x}(x-1) - e^{-x} = e^{-x} \cdot (x - 1 - 1) = e^{-x}(x-2)$

$f''(x) = 0 \Rightarrow x-2=0 \Rightarrow x = 2$

$f''(x) > 0$ für $x > 2$ L

< 0 für $x < 0$ R

e) $x \cdot \ln(x) = f(x) \rightarrow D = \mathbb{R}^+ \Rightarrow x > 0$

$f'(x) = \ln(x) + \frac{x}{x} = \ln(x) + 1$

$f''(x) = \frac{1}{x} = \begin{cases} >0, & x>0 \quad R \\ <0, & x<0 \quad L \end{cases}$

f) $\frac{1}{2}(e^x - e^{-x}) = \sinh(x)$

$f'(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh(x)$

$f''(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh(x)$

$f''(x) = 0 \Rightarrow e^x - e^{-x} = 0 \Rightarrow \underbrace{e^x}_{>0} = \underbrace{e^{-x}}_{>0} \quad | \ln(.)$

$f(x) \Rightarrow x = -x \Rightarrow x = 0$

$f''(x) > 0, \quad x > 0$ L

$f''(x) < 0, \quad x < 0$ R

Blatt 11 Aufgabe 2 (ii)

$\ln(y) = 3x \Rightarrow x = \frac{1}{3}\ln(y)$

$x' = \frac{1}{3y} = \frac{1}{y'}$

$$\int_0^{1/3} \frac{e^{3x}}{e^{3x}+5}\, dx \qquad y = e^{3x} = y(x) \qquad \boxed{\frac{dy}{dx} = 3e^{3x} = 3y}$$

$$= \int_0^{1/3} \frac{e^{3x}}{e^{3x}+5} \underbrace{\frac{dy}{dy}}_{=1} dx = \int_0^{1/3} \frac{e^{3x}}{e^{3x}+5} \underbrace{\frac{dx}{dy}}_{} dy$$

x' für $x = x(y)$

$\left(\frac{1}{\frac{dy}{dx}}\right) = \frac{1}{y'}$

$y(x) \rightarrow dy = y'\, dx \Rightarrow dx = \frac{1}{y'} dy$ $\quad \frac{dx}{dy} = a$

$\frac{1}{a}$

$(y - y_0) = y'(x - x_0)$

$$\frac{1}{3}\int \frac{e^{3x}}{e^{3x}+5}\,\frac{1}{e^{3x}}\,dy = \frac{1}{3}\int_{y(0)}^{y(1/3)} \frac{y}{y+5}\,\frac{1}{y}\,dy$$

$$y = e^{3x}$$

$$= \frac{1}{3}\int_{1}^{e} \frac{1}{y+5}\,dy$$

$$y(0) = e^{3\cdot 0} = 1$$

$$y(1/3) = e^{3/3} = e^{1} = e$$

$$= \frac{1}{3}\ln(|y+5|)\Big|_{1}^{e}$$

$$= \frac{1}{3}\left[\ln(e+5) - \ln(6)\right]$$

$$= \frac{1}{3}\ln\left(\frac{e+5}{6}\right)$$

(iii) a) $I = \int_{2}^{11} \frac{dx}{\sqrt{x-2}}$ $\left(\frac{d}{dx}\ln(\sqrt{x-2}) = \frac{d}{dx}\frac{1}{2}\ln(x-2) = \frac{1}{x-2}\cdot\frac{1}{2} \right)$

$$= 2\sqrt{x-2}$$

Merke: $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

$$I = 2\sqrt{x-2}\Big|_{2}^{11} = 2\sqrt{9} - 2\sqrt{0} = 6$$

b) $\int_{0}^{\infty} 2x e^{-2x}\,dx$

$$\int x e^{-2x}\,dx \overset{p.I}{=} -\frac{1}{2}e^{-2x}\cdot x + \frac{1}{2}\int e^{-2x}\,dx$$

$$= -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}$$

$$= \underbrace{-\frac{e^{-2x}}{2}}\cdot\left(x+\frac{1}{2}\right)$$

Probe: $-\frac{e^{-2x}}{2}\cdot(-2)\cdot(x+1/2) - \frac{e^{-2x}}{2}$

$$= e^{-2x}\cdot(x+\cancel{1/2}) - \cancel{1/2\, e^{-2x}} = x e^{-2x}$$

$$\Rightarrow \int_{0}^{\infty} 2x e^{-2x}\,dx = -\frac{e^{-2x}}{2}\cdot\left(x+\frac{1}{2}\right)\Big|_{0}^{\infty}$$

$$= \lim_{x\to\infty} -\frac{e^{-2x}}{2}\left(x+\tfrac{1}{2}\right) - \lim_{x\to 0} -\frac{e^{-2x}}{2}\left(x+\tfrac{1}{2}\right)$$

$\to 0$ $\quad\quad$ $\downarrow$ $\quad$ $\searrow 0$

$$= \lim_{x\to\infty} \underbrace{-\frac{e^{-2x}}{2}\,x}_{0} + \lim_{x\to\infty} \underbrace{-\frac{e^{-2x}}{2}\cdot\frac{1}{2}}_{0} \qquad \underbrace{e^0 = 1}$$

$x \to \infty$ $\qquad\qquad \to -\frac{1}{2}\cdot\frac{1}{2}$

$$= \frac{1}{4}$$

c) $$I = \int_0^\infty \frac{dx}{(x-1)^3} = \int_0^1 \frac{dx}{(x-1)^3} + \int_1^\infty \frac{dx}{(x-1)^3}$$

$$\int \frac{1}{(x-1)^3}\,dx = -\frac{1}{2}\cdot\frac{1}{(x-1)^2}$$

$$\frac{d}{dx}(x-1)^{-2} = -2\cdot(x-1)^{-2-1} = \underline{-2\cdot(x-1)^{-3}}$$

$$I = -\frac{1}{2}\frac{1}{(x-1)^2}\Bigg|_0^1 - \frac{1}{2}\frac{1}{(x-1)^2}\Bigg|_1^\infty$$

$$= \lim_{x\to 1^-} -\frac{1}{2}\frac{1}{(x-1)^2} - \lim_{x\to 0} \underbrace{-\frac{1}{2}\frac{1}{(x-1)^2}}_{-\frac{1}{2}}$$

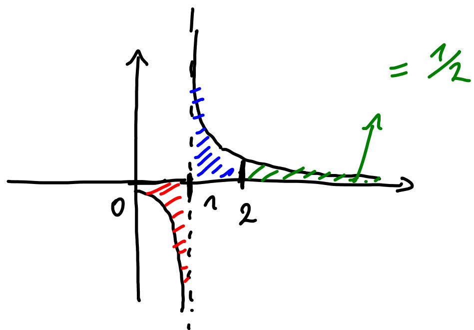
$$+ \lim_{x\to\infty} \underbrace{-\frac{1}{2}\frac{1}{(x-1)^2}}_{0} - \lim_{x\to 1^+} -\frac{1}{2}\frac{1}{(x-1)^2}$$

$$= \lim_{x\to 1^-} -\frac{1}{2}\frac{1}{(\underbrace{x-1}_{<0})^2} - \lim_{x\to 1^+} -\frac{1}{2}\frac{1}{(\underbrace{x-1}_{>0})^2} + \frac{1}{2}$$

$$= \lim_{\varepsilon\to 0^+} -\frac{1}{2}\frac{1}{(-\varepsilon)^2} - \lim_{\varepsilon\to 0^+} -\frac{1}{2}\frac{1}{\varepsilon^2} + \frac{1}{2}$$

$$= \underbrace{\lim_{\varepsilon\to 0^+}\left[-\frac{1}{2}\frac{1}{\varepsilon^2} - \left(-\frac{1}{2}\frac{1}{\varepsilon^2}\right)\right]}_{=0} + \frac{1}{2}$$

$$= +\tfrac{1}{2}$$



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$$\int_0^\infty \frac{dx}{(x-1)^3} \qquad y = x-1 \qquad \frac{dy}{dx} = 1 \Rightarrow dx = dy$$

$$\lim_{x\to\infty} y = \infty$$

$$\int_{\underset{-1}{\overset{\shortparallel}{y(0)}}} \frac{dy}{y^3} = \int_{-1}^{\infty} \frac{dy}{y^3} = \underbrace{\int_{-1}^{1} \frac{dy}{y^3}} + \int_{1}^{\infty} \frac{dy}{y^3}$$

$$f(y) = \frac{1}{y^3} \Rightarrow f(-y) = -f(y)$$

Punktsymmetrie

Blatt 12 Aufgabe 2

a) $$\int \frac{\frac{\sin^2 x}{\cos^4 x} - \frac{4}{\cos^2 x}}{\tan^3 x - \tan^2 x - 4\tan x + 4} \, dx =$$

$$\int \frac{\tan^2 x - 4}{\tan^3 x - \tan^2 x - 4\tan x + 4} \cdot \underbrace{\frac{1}{\cos^2 x}}_{\frac{d \tan x}{dx}} \, dx$$

$$y = \tan x \qquad \frac{dy}{dx} = \frac{1}{\cos^2 x}, \qquad dx = \frac{dx}{dy} dy = \frac{1}{\underbrace{\frac{dy}{dx}}_{\cos^2 x}} dy$$

$$\rightarrow \int \frac{\tan^2 x - 4}{\tan^3 x - \tan^2 x - 4\tan x + 4} \cdot \frac{\cancel{\cos^2 x}^{\,1}}{\cancel{\cos^2 x}} \, dy$$

$$\rightarrow \int \frac{y^2 - 4}{\underbrace{y^3 - y^2 - 4y + 4}_{g(y)}} \, dy \qquad y^2 - 4 = (y+2)(y-2)$$

$g(2) = 8 - 4 - 8 + 4 = 0 \rightarrow y = 2$ Nst $(y-2)$

$$(y^3 - y^2 - 4y + 4) : (y-2) = y^2 + y - 2$$
$$-(y^3 - 2y^2)$$
$$y^2 - 4y + 4$$
$$-(y^2 - 2y)$$
$$-2y + 4 - (-2y+4) = 0$$

$$y^2 + y - 2 = 0 \Rightarrow y = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2}$$

$$y = -\frac{1}{2} \pm \frac{3}{2} \qquad y = 1 \quad \vee \quad y = -2$$

$$g(y) = (y-2)(y+2)(y-1)$$

$$\rightarrow \int \frac{\cancel{(y+2)}\cancel{(y-2)}}{\cancel{(y-2)}\cancel{(y+2)}(y-1)}\, dy = \int \frac{1}{y-1}\, dy$$

$$= \ln(|y-1|)$$

$$= \ln(|\tan x - 1|)$$

b) $$\int_{-1}^{1} \frac{x^2}{\sqrt{1-x}}\, dx$$

$$y = \sqrt{1-x} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{1-x}} \cdot (-1)$$

$$dx = \frac{dx}{dy}\, dy$$

$$\int \frac{x^2}{\sqrt{1-x}}\, dx = -2\int \frac{x^2}{\cancel{\sqrt{1-x}}} \cancel{\sqrt{1-x}}\, dy$$

$$= -2 \int x^2\, dy \qquad x = x(y) = 1 - y^2$$

$$= -2 \int (1-y^2)^2\, dy$$

$$= -2 \cdot \int (1 - 2y^2 + y^4)\, dy$$

$$= -2 \cdot \left(y - \frac{2}{3}y^3 + \frac{1}{5}y^5\right)$$

$$= -2y \cdot \left(1 - \frac{2}{3}y^2 + \frac{1}{5}y^4\right) \qquad y^2 = 1-x$$

$$= -2y \cdot \left(1 - \frac{2}{3}(1-x) + \frac{1}{5}(1-x)^2\right)$$

$$= -\underbrace{2y}_{= \sqrt{1-x}}\left(1 - \frac{2}{3} + \frac{2}{3}x + \frac{1}{5} - \frac{2}{5}x + \frac{1}{5}x^2\right)$$

$$= -2\sqrt{1-x}\left(\frac{1}{3} + \frac{1}{5} + \left(\frac{2}{3} - \frac{2}{5}\right)x + \frac{1}{5}x^2\right)$$

$$\frac{1}{3} + \frac{1}{5} = \frac{5}{15} + \frac{3}{15} = \frac{8}{15}$$

$$\frac{2}{3} - \frac{2}{5} = \frac{10}{15} - \frac{6}{15} = \frac{4}{15}$$

$$= -\frac{2}{15}\sqrt{1-x} \cdot (8 + 4x + 3x^2)$$

$$I = -\frac{2}{15}\sqrt{1-x} \cdot (8 + 4x + 3x^2) \Big|_{-1}^{1}$$

$$= -\frac{2}{15}\underbrace{\sqrt{1-1}}_{0} \cdot (8+4+3)$$

$$+ \frac{2}{15}\sqrt{2} \cdot \underbrace{(8 - 4 + 3)}_{7} = \frac{14}{15}\sqrt{2}$$